SEARCH, ADVERSE SELECTION AND MARKET CLEARING

IN-KOO CHO AND AKIHIKO MATSUI

Abstract. This paper demonstrates that adverse selection can cause the coexistence of involuntary unemployment and involuntary vacancy in a dynamic decentralized trading model, even without search friction. Our model is built on a simple version of Mortensen and Pissarides (1994) with three important changes. First, a seller has private information about the quality of the goods to be traded, while a buyer has to make a decision without observing the true quality (Chang (2012)). Second, the matching technology is efficient in the sense that an agent in the short side of the matching pool is matched with probability 1 and the long side is rationed (Rubinstein and Wolinsky (1985)). Third, to consummate the matching, the two parties must agree upon the price at which the good is delivered. We model the bargaining process as a random proposal model in which a price is randomly drawn by a third party, and each party responds to the price either by accepting or rejecting the offer (Burdett and Wright (1998)). If both parties agree, then the matching is consummated. Otherwise, each party returns to the respective pool, waiting for another round of matching. We quantify the amount of search friction by the time span of each round. We compute a sequence of stationary equilibria as search friction vanishes. We prove that the mass of high quality sellers, the mass of low quality sellers and the mass of buyers are uniformly bounded away from 0, as search friction vanishes.

Keywords: Matching, Search friction, Adverse selection, Undominated equilibrium, Market clearing, Unemployment, Vacancy

> Involuntary unemployment appears to be a persistent feature of many modern labor markets. ... the inability of employers to costlessly observe workers' on-the-job effort, can explain involuntary unemployment as an equilibrium phenomenon (Shapiro and Stiglitz (1984)).

Date: January 8, 2015.

We are grateful for helpful conversation with Asher Wolinsky and Briana Chang. Financial support from the National Science Foundation and the Scientific Research Program (Creative) (19GS0101) and the Grants-in-Aid for Scientific Research (S) (24223002) of the Japan Society for the Promotion of Science is gratefully acknowledged. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation or the Japan Society for the Promotion of Science.

1. Introduction

Persistent coexistence of involuntary unemployment and involuntary vacancy in labor markets is a major challenge to general equilibrium theory. Search theoretic models have been developed to explain this coexistence as an equilibrium outcome in an economy with non-negligible amount of friction. The source of friction can be the time elapsed between two matches for each agent or the inefficiency of matching technology (e.g., Mortensen and Pissarides (1994), Burdett, Shi, and Wright (2001) and Lagos (2000)). This paper demonstrates that adverse selection is another major source of the coexistence, by constructing an equilibrium in a dynamic decentralized trading model, which entails significant amount of "involuntary" unemployment and "involuntary" vacancy, even if search technology is efficient and search friction is arbitrarily small.

In an equilibrium, agents choose, rather than are forced, not to work at the prevailing wage. One can argue that any equilibrium unemployment and vacancy must be voluntary. It is important to clarify our notion of involuntary unemployment and vacancy, and also, the nature of our exercise.

Our notions of involuntary unemployment and vacancy are adopted from Shapiro and Stiglitz (1984). We say that *involuntary unemployment* occurs if the number of people who are willing to work at prevailing prices is greater than the number of people who are actually working. Similarly, we say that *involuntary vacancy* occurs if the number of people whom some other people are willing to hire at prevailing prices is greater than the number of people who are actually hired.[1] Shapiro and Stiglitz (1984) demonstrated that the informational friction arising from moral hazard problem can cause involuntary unemployment in an equilibrium. Yet, Shapiro and Stiglitz (1984) remains silent about whether involuntary vacancy can coexist

[1]Our notion of involuntary unemployment is inspired by Keynes (1936).

> Clearly we do not mean by "involuntary" unemployment the mere existence of an unexhausted capacity to work. ... Nor should we regard as "involuntary" unemployment the withdrawal of their labour by a body of workers because they do not choose to work for less than a certain real reward. Furthermore, it will be convenient to exclude "frictional" unemployment from our definition of "involuntary" unemployment. My definition is, therefore, as follows: *Men are involuntary unemployed if, in the event of a small rise in the price of wage-goods relatively to the money-wage, both the aggregate supply of labour willing to work for the current money-wage and the aggregate demand for it at that wage would be greater than the existing volume of employment.* (page 15, Keynes (1936))

However, we opt for equilibrium analysis of a dynamic model populated by rational agents, rigorously eliminating any source of bounded rationality such as money illusion, or friction such as nominal wage rigidity. (The only source of friction is search friction, but we let it vanish). In this sense, our notion of involuntary unemployment is different from Keynes (1936) as he qualifies it by stating that involuntary unemployment occurs "in the event of a small rise in the price of wage-goods relatively to the money-wage."

with involuntary unemployment, or whether their result can be sustained by a dynamic trading model. We show that the coexistence arises in a dynamic decentralized trading model with adverse selection.

As a benchmark, Akerlof (1970) can be interpreted as modeling a labor market with adverse selection, where the worker knows his own ability (measured in terms of marginal productivity) which affects the output of the firm, whose output price is normalized to 1. One of the main conclusions is that in a competitive equilibrium, only low ability workers will be hired, and high ability workers are unemployed. However, this sort of unemployment is voluntary, not involuntary. Given the prevailing price, it is optimal for a high quality worker to remain unemployed because the prevailing wage is below the reservation wage of the worker. Similarly, given any wage higher than the lowest marginal productivity of a worker, it is optimal for a firm not to hire a worker because the expected value is below the prevailing wage.

The distinctive feature of our equilibrium is that all workers and firms in the market find it optimal to trade at the prevailing price, but have to stay in the market for a significant amount of time. The workers in the market behave "as if" they were involuntarily unemployed, and the firms behave "as if" they were forced to leave the position unfilled. The market fails to clear, even if the matching technology is efficient in the sense that all players in the short side of the labor market are matched with probability one, and even if the search friction is arbitrarily small.

Our main model is built on a simplified version of Mortensen and Pissarides (1994) with three important changes. First, the seller has private information about the quality of the product (Chang (2012)). Second, the matching technology is efficient in the sense that an agent in the short side of the pool is matched with probability 1, while the long side is rationed (Rubinstein and Wolinsky (1985)). Third, in order to consummate the match, the seller and the buyer must agree upon a price at which the good is delivered. We model the bargaining process as a random proposal model (Burdett and Wright (1998)).

An economy is populated by two unit mass of two types of infinitesimal (infinitely-lived) sellers: one unit mass of high quality workers, and another unit mass of low quality sellers. There is an "ocean" of infinitesimal buyers who can freely enter the market. In each period, sellers who know the quality of the good and buyers who do not observe the quality are randomly matched in pairs with a long side being rationed. For each pair, a price is randomly drawn from a continuous density function bounded away from 0. If either party disagrees, then the two agents return to the pool, waiting for another chance to be matched to another agent. If both parties agree, then the trade occurs and the two agents leave the pool of unmatched agents, generating surplus from trading in each period while the agreement is in place.[2] If a buyer remains unmatched, he pays positive vacancy cost. The long term

[2]We choose the random proposal model to ensure that we treat the seller and the buyer equally, while maintaining analytic tractability. If the distribution of the equilibrium payoff is skewed

agreement is dissolved by the decision of either party or by an exogenous shock. Upon dissolution of the long term relationship, the seller returns to the pool of unmatched sellers, but the buyer permanently leaves the market.[3] The objective function of each agent is the expected discounted average payoff.

We focus on a stationary equilibrium in which the agent's decision depends only upon whether or not he is in the long term relationship. We examine stationary equilibria in which trading occurs with a positive probability. In order to crystallize the impact of asymmetric information on involuntary unemployment and vacancy, we examine a sequence of stationary equilibria as the search friction, quantified by the time span of each period, vanishes.

We obtain three main results, in the limit as friction vanishes. First, a positive mass of sellers and a positive mass of buyers are left in the pool in the steady state of a stationary equilibrium. Second, the proportion of sellers of good quality is uniformly bounded away from 0 and 1, which implies that a good quality seller cannot be perfectly sorted out from a bad quality seller. Third, the equilibrium size of involuntary unemployment remains the same even if the amount of involuntary vacancy changes as the vacancy cost varies.

In order to illuminate the relationship between our dynamic model and the static trading model (e.g., Akerlof (1970)), and better reveal the mechanism through which the informational friction leads to the coexistence of unemployment and vacancy, we take a detour. Instead of the main model, we first examine a dynamic auxiliary model, in which the mass of buyers is also fixed, and a buyer pays no vacancy cost, while keeping other elements of the main model the same. After a buyer reaches an agreement with a seller, one unit of goods is delivered to the buyer from the seller at the agreed price as long as the long term contract is in place, as in the main model. However, upon dissolution of the long term relationship, the seller returns to the pool, as the buyer does. That is, a job is not destroyed as in the main model, but recycled in the auxiliary model.

In the auxiliary model, the mass of the buyers and the mass of sellers are fixed, as in Akerlof (1970), which helps us understand how the dynamic trading affects the equilibrium allocation of the static model. Without free entry, a buyer may extract a positive surplus from trading in an equilibrium. By examining the equilibrium payoff of a buyer, we can precisely identify the condition under which the auxiliary model generates an equilibrium where the unemployment and the vacancy coexist. The results of the main model then follow from the analysis of the auxiliary model.

---

toward one party, we can conclude that the result is not a consequence of the difference of the bargaining power which stems from the bargaining rule. The main conclusion is carried over to models with different trading protocols, for example, the one in which one party makes a take-it-or-leave-it offer, and the other party responds by accepting or rejecting the offer. See Cho and Matsui (2013a).

[3]Job destruction is permanent.

In order to illustrate the intuition behind our results, let us consider a market with one unit mass of buyers, and (slightly) less than one unit mass of sellers, each of whom is endowed with one unit of low quality good. In the competitive equilibrium, the short side extracts all the gains from trading and there is no coexistence of involuntary unemployment and vacancy. Cho and Matsui (2012) demonstrated that the same outcome can be sustained by an equilibrium of the auxiliary model without asymmetric information, as friction vanishes.

Next, suppose that we add one unit of high quality sellers, assuming that a buyer does not observe the true quality of the good. As Akerlof (1970) demonstrated, if the average quality in the pool is below the production cost of the high quality good (i.e., "the lemons' problem is severe," which we will assume throughout the paper), then no high quality seller is willing to trade. The static market outcome remains the same as in the model with complete information: only the low quality seller can trade, and the short side extracts the entire gain from trading.

On the other hand, if we add one unit mass of high quality sellers in the auxiliary model, then the static equilibrium outcome is no longer sustained by an equilibrium of the dynamic model. To see this, suppose that the static market outcome as in Akerlof (1970) is sustained by an equilibrium of the dynamic model. If all the low quality sellers reach an agreement and leave the pool, a buyer can infer by the end of the day that virtually every remaining seller has a high quality good. Thus, if a seller is offered a price slightly above the production cost, then buyers and sellers can reach agreement. Contrary to what Akerlof (1970) predicts, a high quality seller may trade at a high price in a dynamic model.

However, a high quality seller cannot reach agreement at a high price "too" frequently in an equilibrium. If trade occur at a high price, then the low quality seller has an incentive to be pooled with the high quality seller to trade at the high price, instead of trading at a low price at which only the low quality seller is willing to trade. If the chance of reaching an agreement at a higher price is "too" high, then every low quality buyer is trying to sell at the higher price. If low quality sellers are completely pooled with the high quality sellers, then due to the assumption that the lemons' problem is severe, the average quality is below the production cost of the high quality good, and trading cannot occur at the high price. Without any signaling mechanism, the high quality sellers cannot completely separate from the low quality seller. As a result, the proportion of the high quality sellers in the pool cannot be 1 or 0, even as the friction vanishes.

In an equilibrium, trading occurs at two prices: one which is so low that only the low quality seller is willing to trade, and the other which both high and low quality sellers are willing to trade.[4] As friction vanishes, a low quality seller's gain from reaching agreement at a lower price over the payoff from staying in the market

[4]More precisely, trading occurs at two separate intervals of prices, each of which converges to a different price as friction vanishes.

vanishes. However, if a low quality seller can trade at a higher price, at which a high quality seller is willing to trade, then the low quality seller's profit over staying in the pool is uniformly bounded away from 0, even as friction vanishes. This gain from pooling a high quality seller provides a strong bargaining position for the low quality seller. As a result, if a low quality seller is negotiating at a lower price, then the seller has a reason to be aggressive.

The probability of reaching agreement is determined by the bargaining position of a seller and a buyer. Unless a buyer is willing to bargain aggressively against an aggressive seller, an agreement can be reached quickly, and the market will clear. Thus, the crucial step to identify the condition under which the market fails to clear even in the limit as friction vanishes is to identify a condition under which the buyer is bargaining aggressively. A buyer who has no informational rent will bargain aggressively if the delay is not costly, which is the case if the equilibrium payoff is vanishingly small. In the main model, the free entry condition forces the equilibrium payoff of a buyer to be 0. As a result, the trading between a low quality seller and a buyer is realized so slowly that a positive mass of buyers and a positive mass of low quality sellers must remain in the market for an extended period of time. Since the proportion of the low quality seller cannot be 1 or 0, a positive mass of high quality sellers are also staying in the pool for a significant amount of time. As a seller and a buyer reach agreement too slowly, a positive mass of buyers and sellers have to stay in the pool for a while, even though both parties are willing to trade at a high price. Involuntary unemployment and vacancy arise, because of adverse selection and dynamic trading, without search friction.

The low quality seller's bargaining power arises from the possibility that if the current bargaining falls apart, he can immediately meet another buyer. If the low quality seller is in the short side of the market, then he will be matched to another buyer one period after the present bargaining falls apart. Suppose that there is a positive mass of buyers in the pool, and the seller is in the long side of the market. Since the seller is rationed, the probability that a seller is matched to a buyer in one round is less than one. However, for any time interval, the number of matches increases as the time span per period converges to zero, and the probability for a seller to meet a buyer within a positive time interval converges to one. Thus, the low quality seller's bargaining power is not affected by how many buyers are in the pool, as long as a positive mass of buyers is in the pool. As a result, the same mass of sellers are left out in the pool as in the case where the low quality seller is in the short side of the market.

Literature on markets with adverse selection in a search model is extensive. Before describing our model formally, Section 2 illustrates the important features of our model and the key difference of our questions from what existing models investigate. Section 3 describes the model in which the masses of sellers and buyers are exogenously given. Section 4 presents the preliminary results and concepts. Section

5 formally describes the main results. Section 6 considers a model in which true quality of the good is revealed dring the long term relationship. Section 7 concludes the paper.

2. Literature Review and Questions

2.1. **Inefficient matching and adverse selection.** Adverse selection in a search model has recently drawn considerable attention. Guerrieri, Shimer, and Wright (2010) and Chang (2012) examined a class of matching models with adverse selection. A key assumption is that the matching technology is not efficient, in the sense that the short side in the matching pool may have a limited opportunity to meet a partner. Because both the inefficiency of matching technology and the informational asymmetry contribute to unemployment and vacancy, it is difficult to identify the size of impact on the equilibrium outcome by the informational asymmetry from the unemployment caused by inefficiency of matching technology. It is essential to assume an efficient matching technology so that the only source of friction other than informational asymmetry is the time span of each period, which we let vanish. We chose the random matching as an example of an efficient matching technology, but the main conclusion continues to hold, as long as the matching technology is efficient.

2.2. **One-sided involuntary unemployment.** Diamond (1971) is followed by a large number of papers, showing under various informational and institutional assumption that involuntary unemployment can persist. In Diamond (1971), a positive amount of excess supply persists in the equilibrium, but a buyer can purchase a good without any constraint.

Shapiro and Stiglitz (1984) constructed a model in which, in the presence of moral hazard, the wage rate is set higher than the market clearing wage to induce "involuntary" unemployment of workers. This excess supply of labor is needed to create an incentive for the workers to make an effort in fear of being fired and unemployed for a while. It is implicitly assumed that the firm can fill the vacancy at the prevailing wage.

Stiglitz and Weiss (1981) examined a credit market in which the excess supply of the credit persists, and the credit must be rationed in an equilibrium. The adverse selection problem on the part of the borrowers keeps the interest rate low, which forces the credit to be rationed in an equilibrium. Azariadis (1975) studied an industry with demand uncertainty that prompts risk-neutral firms to act both as employers and as insurers of risk-averse workers. Firms may lay off, by random choice, part of the work force when the demand is in a low state.

2.3. **Steady state and inefficiency.** The goal of this paper is *not* to examine whether an equilibrium allocation of a decentralized trading model is inefficient, which has been the focus of virtually all existing papers on the matching model with adverse selection. Instead, our objective is to show that the market can fail to clear in a steady state of a dynamic decentralized trading model under adverse selection, even if friction vanishes.[5] No existing model of dynamic decentralized trading is designed to answer this question.

It is well known that asymmetric information can cause significant delay in bargaining, even in the limit as friction vanishes (e.g., Vincent (1989) and Fuchs and Skrzypacz (2013)), and in a decentralized trading model in which traders leave the economy permanently after reaching agreement (e.g., Blouin and Serrano (2001)). In this class of models, the mass of traders in the economy decreases over time, and the market must clear in the long run.

It is significantly more challenging to show that asymmetric information can cause inefficiency in a steady state of a decentralized trading models, which is a necessary but not a sufficient condition for the failure of market clearing. While asymmetric information about private valaution may cause delay in bargaining (e.g., Ausubel and Deneckere (1989)), the decentralized trading mechanism can aggregate the dispersed private information to achieve efficient allocation (e.g., Satterthwaite and Shneyerov (2007)). Asymmetric information about the common value component, which is the foundation of adverse selection problem, is known to generate inefficient allocation in a steady state of a dynamic decentralized trading model (e.g., Moreno and Wooders (2010), Kim (2012) and Lauermann and Wolinsky (2013)).

Matsui and Shimizu (2005) examined the coordination failure among agents, who are searching a particular post to trade, out of many posts. But, Matsui and Shimizu (2005) proves that the coordination failure can be resilient: the coordination failure may not vanish even in the limit as the friction vanishes. As a result, some post can experience a positive amount of unemployment, while some other experiences a positive amount of vacancy, which could have been avoided if the agents could have coordinated their search for the trading posts.

Most existing models of decentralized dynamic trading assume that a fresh flow of agents enter the market as a group of traders leave immediately after reaching agreement in order to keep the size of the matching pool bounded away from 0. With continuous inflow of fresh agents, the total supply and the total demand in the economy is infinite. As a result, any finite amount of unemployment or vacancy is insigificant. It is essential to fix the total amount of supply or demand.[6] To this end, we examine a "closed" system in which no agent leaves the economy. After a buyer and a seller reach an agreement, they leave the market but stay in the economy, accruing surplus until the long term agreement dissolves by an exogenous shock with a small probability. We shall show that the condition under which the bargaining entails a significant delay does not guarantee that the market fails to

[5]The inefficiency of the equilibrium allocation follows as an easy corollary.

[6]In a typical labor market search model, the total supply of labor is fixed (e.g., Mortensen and Pissarides (1994).

clear, in the limit as friction vanishes. We then extend the result to the case in which buyers can enter the economy freely, but pay a vacancy cost while in the matching pool, while the supply of the labor is fixed.

2.4. **Mechanism design approach.** A two person bargaining problem under adverse selection (Samuelson (1984)) with a continous type space can be viewed as a trading model between a mass of buyers and the same mass of sellers who have private information on the quality of the product. Samuelson (1984) extends and deepens the insight of Akerlof (1970), characterizing all incentive compatible mechanisms including the one corresponding to the outcome of Akerlof (1970). Samuelson (1984) shows that under a very general condition, adverse seletion can cause inefficient allocation, or even no trading in equilibrium.

While the interim efficient mechanism is a focal point of the analysis of Samuelson (1984), the same paper does not spell out a formal criterion to select a particular incentive compatible mechanism. Our exercise can be viewed as a selection of a most plausible mechanism out of many incentive compatible mechanisms, by emulating an incentive compatible mechanism outcome by a sequence of "simple" equilibria in "sensible" dynamic trading models.[7] To this end, we focus on a simple equilibrium (i.e., stationary equilibrium) in a canonical decentralized dynamic trading model.

3. Model

3.1. **Static model.** We consider an economy which is populated by 2 unit mass of infinitesimal (infinitely-lived) sellers, high type and low type sellers of equal size, and $x_b > 0$ unit mass of infinitesimal (infinitely-lived) buyers.[8]

High type sellers produce one unit of high quality good at the cost of c_h, while low type sellers produce one unit of low quality good at the cost of c_l. Assume $c_h > c_l$. The goods are indivisible. The marginal utility of the high quality good for a buyer is ϕ_h, while that of the low quality good is ϕ_l, where $\phi_h > \phi_l$. Each seller produces at most one unit of the good, and each buyer consumes at most one unit of the good.

We make the following three standard assumptions on the parameter values, which are critical for capturing the lemons problem.

*A*1. $\phi_h > c_h > \phi_l > c_l$, which implies that the existence of the gains from trading under each state is common knowledge.

[7]Cho and Matsui (2013b) selects a particular competitive equilibrium outcome, if there are multiple equilibrium prices. One can view Lauermann and Wolinsky (2013) as the same kind of a mechanism selection exercise, as they focus on a particular incentive compatible mechanism in Samuelson (1984) which can be approximated by a sequential equilibrium in a dynamic trading model.

[8]No main result is qualitatively sensitive to the fact that the masses of high and low quality sellers are the same.

*A*2. $\phi_h - c_h > \phi_l - c_l$ so that it is socially efficient for the high quality sellers to deliver the good to the buyers.

*A*3. $\dfrac{\phi_h + \phi_l}{2} < c_h$ so that the lemons problem is severe in the sense that random transactions lead to a negative payoff either to a buyer or to a high quality seller.

If p is the delivery price of the good, and $y(\in \{h, l\})$ is the quality of the good, seller's profit is $p - c_y$ and buyer's surplus is $\phi_y - p$. Under the assumptions we made, only the low quality good is traded in any competitive equilibrium, and the equilibrium price p^* is given by

$$p^* \in \begin{cases} \{c_l\} & \text{if } x_b < 1, \\ [c_l, \phi_l] & \text{if } x_b = 1, \\ \{\phi_l\} & \text{if } x_b > 1. \end{cases}$$

3.2. **Dynamic auxiliary model.** Let us embed the above static model into a decentralized dynamic trading model. We first describe the auxiliary model, where the mass of buyers is exogenously fixed to $x_b > 0$, as in the static model.

Time is discrete, and the horizon is infinite. The time span of each period is $\Delta > 0$, which represents the amount of friction. When a buyer and a seller are initially matched at period t, conditioned on her type $k \in \{h, l\}$, the seller reports her type as $k' \in \{1, 2\}$, possibly in a randomized fashion, to a third party (or mechanism) which draws a price p according to a probability density function $f_{k'}$ over $\mathbb{R}$. We assume that the support of $f_{k'}$ is $[c_l, \phi_h]$.

We assume

$$\forall k' \in \{h, l\},\ \forall p \in [c_l, \phi_h], \quad f_{k'}(p) > 0 \text{ and is continuous.} \tag{3.1}$$

Conditioned on p drawn by the mechanism, each party has to decide whether or not to form a long term relationship. After forming the long term relationship, the buyer can purchase the good at the agreed price, and the seller can sell the good at the same price to the buyer. If the good is delivered at p, the seller's surplus is $p - c_k$ and the buyer's surplus is $\phi_k - p$ $(k \in \{h, l\})$.

Then, at the end of the period, either one of two events will occur. The long term relationship breaks down with probability $1 - \delta$, and then, both agents are dumped back to the respective pools. The long term relationship continues with probability δ without the true quality being revealed.

In each period, the buyer and the seller in a long term relationship can choose to maintain or to terminate it. If one of the agents decides to terminate the long term relationship, both agents return to their respective pools, waiting for the next round of matching. If both agents decide to continue the long term relationship, the long term relationship continues with probability $\delta = e^{-d\Delta}$ where $d > 0$, and with probability $1 - \delta$, the long term relationship dissolves, and the two agents are forced to return to the pool.

We assume that the true quality of the good is not revealed to the buyer during the long term relationship, like a life insurance policy, until the long term relationship dissolves. This assumption is only to simplify exposition.[9]

The objective function of each agent is the long run discounted average expected payoff:

$$(1-\beta)\mathsf{E}\sum_{t=1}^{\infty}\beta^{t-1}u_{i,t}$$

where $u_{i,t}$ is the payoff of agent i in period t and $\beta = e^{-b\Delta}$ is the discount factor.

We focus on a simple class of equilibria, in which the equilibrium strategy of each player depends only upon the status of the player: whether or not the player is in the long term relationship. A stationary equilibrium is a strategy profile where no player has an incentive to deviate, and a distribution of the agents in the pool does not change over time. Like many other bilateral trading models, this model admits a stationary equilibrium with no trading, as every player rejects every price following every history. We focus on the undominated stationary equilibrium, which is a stationary equilibrium where no dominated strategy is used, to exclude a "no trading equilibrium" in which every agent refuses to reach an agreement. We simply refer to an undominated stationary equilibrium as an equilibrium, whenever the meaning is clear from the context.

To simplify exposition, we assume for the rest of the paper that p is drawn from $[c_l, \phi_h]$ according to the uniform distribution regardless of the report of the seller. We can assume that p is drawn from $[c_l, \phi_h]$ according to a continuous density function $f(p) > 0\ \forall p \in [c_l, \phi_h]$, without changing any result, but only at the cost of significantly more cumbersome notation.[10]

3.3. **Main model.** While leaving all other elements of the auxiliary model intact, we modify two elements of the auxiliary model to build the main model: free entry by the buyers, and job destruction, as in Mortensen and Pissarides (1994). First, we assume free entry of buyers to the matching pool, instead of a fixed mass of buyers. To stay in the matching pool, a buyer has to pay vacancy cost $F > 0$ per unit of time, i.e., ΔF per period. If a buyer and a seller do not agree to form a long term relationship, each party remains in their respective pool, waiting for the next round of matching.

Second, we assume that the job is destroyed rather than recycled as in the auxiliary model. If a buyer and a seller reach agreement, they leave the pool. While the long term contract is in place, a seller delivers one unit of the good at the agreed

[9] In subsection 6, we extend the model to the one in which the true quality is revealed to the buyer during the long term relationship with probability $1-\lambda$ per period, and upon the revelation of the true quality, the buyer can decide to continue or terminate the existing long term relationship.

[10] The extension to the case where the price is drawn from a general distribution satisfying (3.1) is (Cho and Matsui (2013b)).

price in each period. If the existing long term relationship dissolves voluntarily or involuntarily, the buyer leaves the economy permanently, while the seller returns to the pool of unmatched sellers.

As in the auxiliary model, we focus on a stationary equilibrium with a positive probability of trading. Due to the free entry condition, a buyer's long run average payoff is zero, in any equilibrium.

4. Preliminaries

We analyze the auxiliary model for the most part of the paper, which better reveals how the equilibrium payoff of the buyer affects the market clearing condition than the main model. After completely analyzing the auxiliary model, we derive main results of the main model.[11]

Let $W_s^h(p)$, $W_s^l(p)$, and $W_b(p)$ be the continuation values of a high quality seller, a low quality seller, and a buyer, respectively, after the two agents agree on $p \in [c_l, \phi_h]$. Also, let W_s^h, W_s^l, and W_b be the continuation values of respective agents after they do not form a long term relationship. Given the equilibrium value functions, let us characterize the optimal decision rule of each agent. In what follows, we write $x \leq O(\Delta)$ if

$$\lim_{\Delta \to 0} \frac{x}{\Delta} < \infty.$$

Let z_s^l and z_s^h be the mass of c_l and c_h sellers in the pool. Similarly, let z_b be the mass of buyers in the pool. Since the mass of paired buyers and the mass of paired sellers are of equal size, we have

$$2 - z_s = x_b - z_b, \tag{4.2}$$

where $z_s = z_s^h + z_s^l$. Let

$$\mu_h = \frac{z_s^h}{z_s}$$

be the proportion of high quality sellers in the pool of sellers, and let $\mu_l = 1 - \mu_h$ be the proportion of low quality sellers in the pool of sellers.

Our first goal is to find conditions under which

$$\lim_{\Delta \to 0} z_b > 0,$$

and

$$\lim_{\Delta \to 0} z_s > 0$$

hold simultaneously in the auxiliary model, where the mass of buyers is fixed. Throughout the paper, z_s is interpreted as (involuntary) unemployment, while z_b as (involuntary) vacancy.

[11]Most notation and concepts defined for the auxiliary model will be used for the main model, after a minimal number of necessary changes.

Because the relative size of buyers and sellers in the pool is an important variable, let us define

$$\rho_{bs} = \frac{z_b}{z_s}.$$

Since ρ_{bs} determines the frequency of meeting the other party with a long side rationed, let us define

$$\zeta = \min\{1, \rho_{bs}\}$$

as the probability that a seller meets a buyer, and

$$\xi = \min\left\{1, \frac{1}{\rho_{bs}}\right\} \tag{4.3}$$

as the probability that a buyer meets a seller, where we treat $1/0 = \infty$. Due to (4.2), we have

$$\begin{cases} \zeta = \rho_{bs} < 1 \text{ and } \xi = 1 & \text{if } x_b < 2, \\ \zeta = \rho_{bs} = 1 \text{ and } \xi = 1 & \text{if } x_b = 2, \\ \zeta = 1 \text{ and } \xi = \frac{1}{\rho_{bs}} < 1 & \text{if } x_b > 2. \end{cases}$$

Let Π_s^h be the set of prices that a high quality seller and a buyer agree to accept, and let $\pi_s^h = \mathsf{P}(\Pi_s^h)$. For $p \in \Pi_s^h$, we can write

$$W_s^h(p) = (1-\beta)(p - c_h) + \beta\left(\delta W_s^h(p) + (1-\delta)W_s^h\right).$$

The first term is the payoff in the present period. At the end of the present period, with probability $1 - \delta$, the long term relationship dissolves, and the high quality seller's continuation payoff is W_s^h. With probability δ, the high quality seller continues the relationship, of which continuation value is given by $W_s^h(p)$.

A simple calculation shows

$$W_s^h(p) = \frac{(1-\beta)(p-c_h) + \beta(1-\delta)W_s^h}{1-\beta\delta}. \tag{4.4}$$

The high quality seller agrees to form a long term relationship with delivery price p if

$$W_s^h(p) > W_s^h$$

which is equivalent to

$$p > c_h + W_s^h. \tag{4.5}$$

On the other hand, W_s^h is given by

$$W_s^h = \beta\zeta\pi_s^h \mathsf{E}[W_s^h(p)|\Pi_s^h] + \beta(1-\zeta\pi_s^h)W_s^h. \tag{4.6}$$

Substituting (4.4) into (4.6), we obtain, after some calculation,

$$W_s^h = \frac{\beta\zeta\pi_s^h}{1-\beta\delta}\mathsf{E}[p - c_h - W_s^h|\Pi_s^h]. \tag{4.7}$$

Similarly, we obtain

$$W_s^l = \frac{\beta\zeta\pi_s^l}{1-\beta\delta}\mathsf{E}[p - c_l - W_s^l | \Pi_s^l], \tag{4.8}$$

where Π_s^l is the set of prices that a low quality seller and a buyer agree to accept, and $\pi_s^l = \mathsf{P}(\Pi_s^l)$. In any undominated equilibrium, c_l seller accept p if

$$p > c_l + W_s^l.$$

Imitating the behavior of high quality sellers, a low quality seller can always obtain a higher (or equal) continuation value than a high quality seller.[12] Therefore, we have $W_s^l \geq W_s^h$. Now, we would like to claim that the threshold price for a low quality seller is lower than that for a high quality seller.

Lemma 4.1.

$$c_h - c_l > W_s^l - W_s^h.$$

Proof. If a high quality seller imitates a low quality seller, then the long run expected payoff from the deviation is

$$W_s^l - (c_h - c_l)\frac{\beta\pi_s^l}{1-\beta\delta+\beta\pi_s^l}.$$

Since the deviation payoff is less than the equilibrium payoff,

$$W_s^l - W_s^h \leq (c_h - c_l)\frac{\beta\pi_s^l}{1-\beta\delta+\beta\pi_s^l} < c_h - c_l$$

as desired. □

Let Π_s^l (resp. Π_s^h) be the set of prices where L-type (resp. H-type) sellers and buyers trade with a positive probability. Lemma 4.1 says

$$c_l + W_s^l = \inf \Pi_s^l < c_h + W_s^h = \inf \Pi_s^h.$$

Since the decision rule of each seller is a threshold rule, this inequality implies

$$\Pi_s^h \subset \Pi_s^l.$$

Thus, we can partition the set of prices into three regions, Π_s, Π_p, and the rest:

$$\begin{aligned}\Pi_s &= \Pi_s^l \setminus \Pi_s^h,\\ \Pi_p &= \Pi_s^l \cap \Pi_s^h,\end{aligned}$$

where Π_s is the set of the prices at which trade occurs only with low quality sellers (the subscript stands for separating), Π_p is the set of the prices at which trade occurs

[12]If the true quality is revealed with a positive probability after the good is delivered, then we cannot invoke the same argument to prove the inequality. Yet, the main result is carried over.

with both low and high quality sellers (the subscript stands for pooling), and the remaining region is the one in which no trade occurs. Note that we have

$$\begin{aligned}\Pi_s &\subset [c_l + W_s^l, c_h + W_s^h],\\ \Pi_p &\subset [c_h + W_s^h, \infty).\end{aligned}$$

We shall focus on a stationary equilibrium in which the strategy of each player depends only upon the status of the player, i.e., whether or not he is in the pool or in the long term relationship. As in most bilateral trading models, this game admits a stationary equilibrium, in which each player rejects all prices, following every history. In this equilibrium, also known as no trading equilibrium, some players have to use (weakly) dominated strategies. Note that the equilibrium payoff is 0 for every player. Under $A1$, any price $p \in (c_l, \phi_l)$ is accepted, if all, only by the low quality seller. If the buyer accepts such p, he can still generate strictly positive surplus. Only because each player reject such p with probability 1, it is optimal to reject p. If p is accepted by the other party with a positive probability, it is the best response to accept p. Thus, rejecting p is a (weakly) dominated strategy.

Eliminating no trading equilibrium, we focus on a stationary equilibrium in which trading occurs with a positive probability. Let $\pi_s = \mathsf{P}(\Pi_s)$ and $\pi_p = \mathsf{P}(\Pi_p)$. Since we focus on an equilibrium in which trading occurs with a positive probability,

$$\pi_s + \pi_p > 0$$

in an equilibrium. Since we eliminate no trading equilibrium, which involves (weakly) dominated strategies, we call our equilibrium an undominated stationary equilibrium, or simply, an equilibrium, whenever the meaning is clear from the context.

Definition 4.2. *If $\pi_p = 0$ in an equilibrium, we call such an equilibrium a separating equilibrium. If $\pi_s = 0$, then the equilibrium is called a pooling equilibrium. If $\pi_s > 0$ and $\pi_p > 0$, then it is called a semi-pooling equilibrium.*

Let us calculate the value function of a buyer. In the private value model in which a buyer knows exactly how valuable the objective is (Cho and Matsui (2012)), the informational content of p is irrelevant for a buyer to deciding whether or not to accept p. In contrast, in the present model, the expected quality conditioned on p is a critical factor for a buyer to make a decision.[13] Let $\phi^e(p)$ be the expected quality if p is the price to be agreed upon. If $p \in (c_l + W_s^l, c_h + W_s^h)$, then only low quality sellers agree to accept the price, and therefore, we have $\phi^e(p) = \phi_l$. On the other hand, if $p > c_h + W_s^h$ holds, then both low and high quality sellers agree to do so, and therefore, we have

$$\phi^e(p) = \phi(\mu_l) \equiv \mu_l\phi_l + (1 - \mu_l)\phi_h.$$

[13]Even if each individual is infinitesimally small, the informational content of p affects the decision of all buyers. In this sense, each individual is not "informationally small" in the sense of Gul and Postlewaite (1992).

If a buyer and a seller agree to form a long term relationship at price p, then the expected continuation value of the buyer is given by

$$W_b(p) = (1-\beta)(\phi^e(p) - p) + \beta\left[\delta W_b(p) + (1-\delta)W_b\right].$$

Therefore, we have

$$W_b(p) = \frac{(1-\beta)(\phi^e(p) - p) + \beta(1-\delta)W_b}{1-\beta\delta}.$$

Also, the continuation value after no match is given by

$$W_b = \beta\xi\mu_l\pi_s \mathsf{E}\left[W_b(p)|\Pi_s\right] + \beta\xi\pi_p\mathsf{E}\left[W_b(p)|\Pi_p\right] + \beta(1-\xi\mu_l\pi_s - \xi\pi_p)W_b.$$

After substitutions and tedious calculation, we obtain

$$W_b = \frac{\beta\xi\mu_l\pi_s}{1-\beta\delta}\mathsf{E}\left[\phi_l - p - W_b|\Pi_s\right] + \frac{\beta\xi\pi_p}{1-\beta\delta}\mathsf{E}\left[\phi(\mu_l) - p - W_b|\Pi_p\right] \tag{4.9}$$

where ξ is the probability that a buyer is matched to a seller, as defined in(4.3).

A buyer is willing to accept p if

$$W_b(p) > W_b,$$

or equivalently,

$$\phi^e(p) - p > W_b.$$

Since $\phi^e(p)$ may change as p changes, the buyer's equilibrium decision rule may not be characterized by a single threshold.

Combining these results and including the endpoints as they are measure zero events, we have

$$\Pi_s = \begin{cases} [c_l + W_s^l, \phi_l - W_b] & \text{if } c_l + W_s^l \le \phi_l - W_b, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\Pi_p = \begin{cases} [c_h + W_s^h, \phi(\mu_l) - W_b] & \text{if } c_h + W_s^h \le \phi(\mu_l) - W_b, \\ \emptyset & \text{otherwise.} \end{cases}$$

By the assumption that p is uniformly distributed over (c_l, ϕ_h), we obtain

$$\mathsf{E}[p - c_h - W_s^h|\Pi_s^h] = \mathsf{E}[p - c_h - W_s^h|\Pi_p] = A\pi_p,$$

where

$$A = \frac{1}{2}(\phi_h - c_l).$$

Therefore, (4.7) can be rewritten as

$$W_s^h = \frac{\beta A(\pi_p)^2\zeta}{1-\beta\delta}. \tag{4.10}$$

Similarly, we have

$$\mathsf{E}[p - c_l - W_s^l|\Pi_s] = A\pi_s, \tag{4.11}$$

$$\mathsf{E}[\phi(\mu_l) - p - W_b|\Pi_p] = A\pi_p, \tag{4.12}$$

$$\mathsf{E}[\phi_l - p - W_b|\Pi_s] = A\pi_s. \tag{4.13}$$

Thus, W_s^l and W_b can be rewritten as

$$W_s^l = \frac{\beta A(\pi_s)^2\zeta}{1-\beta\delta} + \frac{\beta\pi_p\zeta}{1-\beta\delta}\mathsf{E}[p - c_l - W_s^l|\Pi_p] \tag{4.14}$$

$$W_b = \frac{\beta A(\pi_s)^2\mu_l\xi}{1-\beta\delta} + \frac{\beta A(\pi_p)^2\xi}{1-\beta\delta}, \tag{4.15}$$

respectively. Following Cho and Matsui (2013b), one can obtain a similar expression for a general distribution of price, given a sufficiently small $\Delta > 0$.

Also, rewrite π_s and π_p as

$$\pi_s = C[\phi_l - c_l - W_b - W_s^l] \tag{4.16}$$

$$\pi_p = C[\phi(\mu_l) - c_h - W_b - W_s^h] \tag{4.17}$$

where

$$C = \frac{1}{\phi_h - c_l}.$$

The size of population of each type of the agents is determined by the balance equations:

$$1 - z_s^l = \left(\frac{\pi_s\zeta}{1-\delta} + \frac{\pi_p\zeta}{1-\delta}\right) z_s^l \tag{4.18}$$

$$1 - z_s^h = \frac{\pi_p\zeta}{1-\delta} z_s^h \tag{4.19}$$

$$x_b - z_b = \left(\frac{\pi_s\mu_l\xi}{1-\delta} + \frac{\pi_p\xi}{1-\delta}\right) z_b. \tag{4.20}$$

An equilibrium in the main model is characterized by $(z_b, z_s^h, z_s^l, W_b, W_s^l, W_s^h)$. We use the following notion of market clearing.

Definition 4.3. *A market fails to clear if (involuntary) unemployment and (involuntary) vacancy coexist:*

$$\lim_{\Delta\to 0} z_b z_s = \lim_{\Delta\to 0} z_b(z_s^h + z_s^l) > 0. \tag{4.21}$$

Otherwise, the market clears.

What we would like to show is that in the main model, the market fails to clear.

5. Analysis

Since the main result takes a number of steps, it will be helpful to illustrate the reasoning process toward the main result.

5.1. **Overview.** Note that z_b, z_s^h, z_s^l are functions of π_s and π_p. The smaller π_s and π_p are, the larger z_b, z_s^h, z_s^l. As $\Delta \to 0$, each player has more opportunities to meet his potential partner for a given amount of (real) time. The ensuing analysis shows that $\pi_s, \pi_p \to 0$ as $\Delta \to 0$.

In order to identify the condition under which (4.21) occurs, we need to analyze how quickly $\pi_s, \pi_p \to 0$ as $\Delta \to 0$. To be concrete, suppose that π_s, π_p vanishes at a slower rate than Δ. Then, for $\forall \tau > 0$ amount of real time, the agreeable prices arrive quickly so that all opportunities to trade will be exhausted as $\Delta \to 0$. If so, the market must clear. On the other hand, if π_s, π_p vanishes at the rate of Δ, then the trading occurs slowly enough so that some traders have to remain in the pool for a significant amount of time before reaching agreement. In fact, we are looking for the condition in the auxiliary model where $\pi_s, \pi_p \to 0$ at the rate of $\Delta > 0$.

The question is, then, what causes $\pi_s, \pi_p \to 0$ "quickly." Remember that $\Pi_s = [c_l + W_s^l, \phi_l - W_b]$ and $\Pi_p = [c_h + W_s^h, \phi^e - W_b]$. The lower bound of Π_s is determined by the equilibrium threshold price of c_l seller, while its upper bound is the equilibrium threshold price of a buyer. The size of Π_s, and π_s is therefore determined by how aggressively each party bargains. If a c_l seller bargains aggressively, $c_l + W_s^l$ will be set high, and if a buyer bargains aggressively, $\phi_l - W_b$ will be set low.

The first crucial step is to identify a player who bargains aggressively, and the source of his bargaining power. Note that c_l seller has an option to sell at a price in Π_s, but also at a price in Π_p. If trading occurs at a price in Π_s, the gain from trading over W_s^l is small, since Π_s is shrinking to a single point as $\Delta \to 0$. However, if trading occurs at a price Π_p, a c_l seller can generate at least $c_h - \phi_l > 0$. Thus, c_l seller is bargaining very aggressively when $p \in \Pi_s$ is drawn.

If the buyer is accommodating the aggressive demand of c_l seller, then trading can occur in Π_s frequently to clear the market. The next crucial step is to identify the condition under which a buyer is also bargaining aggressively. A buyer is willing to yield to the demand of a seller, if delay is costly, which is the case if $\lim_{\Delta \to 0} W_b > 0$. If $W_b \to 0$ (or $W_b = 0$ as in the main model), however, a buyer has little to lose by delaying the agreement. Theorem 5.5 completely characterizes the condition under which $\lim_{\Delta \to 0} W_b = 0$ in the auxiliary model under which the market fails to clear. Since $W_b = 0$ in the main model, it is verified that the market fails to clear in the main model.

5.2. **Results from the auxiliary model.** Let us state the asymptotic properties of the equilibrium payoffs for the case where $A1 - A3$ hold, in the auxiliary model.

Proposition 5.1. *For any sequence of undominated stationary equilibria,*

$$\begin{aligned}\lim_{\Delta\to 0} W_s^h &= 0\\ \lim_{\Delta\to 0} W_s^l + W_b &= \phi_l - c_l\end{aligned}$$

Proof. See Appendix A. □

In order to understand how the equilibrium surplus $\phi_l - c_l$ is split between a seller and a buyer, we need to investigate the structure of an equilibrium further. The next lemma is a critical step toward characterizing the condition under which the market fails to clear.

Lemma 5.2.

$$\lim_{\Delta\to 0} \zeta W_b = 0.$$

Proof. See Appendix B. □

The alternative characterization of a Nash bargaining solution by Harsanyi (1956) offers a useful insight toward Lemma 5.2. Let us imagine a bargaining situation between a low quality seller and a buyer, and assume for a moment that $p < c_h$ is on the table.[14] The option value of rejecting the present offer is roughly the future gain multiplied by the probability of reaching an agreement. Alluding to Harsanyi (1956), we may claim that the bargaining power can be represented as a ratio of the option value of rejecting the current offer over the present value of accepting it. The bargaining outcome is determined by equating the bargaining powers of the two parties. Equivalently, the ratio between the equilibrium payoffs of the two parties must be equal to the ratio between the option values of each party for rejecting the present offer in an equilibrium. The precise ratio of the option values is given by

$$\frac{A\zeta\pi_s^2 + \zeta\pi_p \mathsf{E}[p - c_l - W_s^l | \Pi_p]}{A\left(\mu_l \pi_s^2 + \pi_p^2\right)}.$$

In equilibrium, this must be equal to W_s^l / W_b.

Note that the low quality sellers trade either at $p \in \Pi_s$, or at $p \in \Pi_p = [c_h + W_s^h, \phi^e - W_b]$. The net gain from reaching agreement over staying in the pool is $p - c_l - W_s^l > 0$. If p is drawn from $\Pi_p = [c_h + W_s^h, \phi^e - W_b]$, then the net gain from trading at a price $p \in \Pi_p$ does not vanish even if $\Delta \to 0$:

$$p - c_l - W_s^l \geq c_h - c_l - W_s^l = (c_h - \phi_l) + (\phi_l - c_l - W_s^l) \to (c_h - \phi_l) + \lim_{\Delta\to 0} W_b \geq c_h - \phi_l > 0$$

by Proposition 5.1, while the last (strict) inequality follows from $A1$.

Suppose that $\zeta > 0$ is uniformly bounded away from 0 as $\Delta \to 0$. The low quality seller has an opportunity to make a large profit if she can sell the good at a price drawn from Π_p. However, this profit can be realized, only if she can be matched to a

[14] The buyer knows only the low quality seller will accept such a price.

buyer. On the other hand, the buyer does not have an opportunity to make as large a profit as the seller, because he has no private information. If $\zeta > 0$ is bounded away from 0, a low quality seller has a positive chance to realize a large profit from trading at a price drawn from Π_p. The ratio between the option values of rejecting the present price between the buyer and the low quality seller converges to 0. In an equilibrium, this ratio must be the same as the ratio of the equilibrium payoffs of the buyer and the low quality seller. Therefore, $W_b \to 0$ as $\Delta \to 0$.

Lemma 5.2 reveals the complementary slackness between z_b (or ζ), and W_b in the limit as $\Delta \to 0$. If $\lim_{\Delta\to 0} z_b > 0$, then the buyer's long run average payoff must vanish. In the private value model, in which the private valuation of the good is private information, this result is a consequence of the competition among buyers in the market (Lauermann (2013)). In our case, where the buyer does not have private information about the common value component, this result is a consequence of the aggressive bargaining by the low quality seller, as the ensuing analysis shows.

Proposition 5.3. *If*

$$\lim_{\Delta\to 0} z_b > 0,$$

then

$$\lim_{\Delta\to 0} z_s = \frac{\phi_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)} < 2. \tag{5.22}$$

Proof. From (4.18) and (4.19), we have

$$z_s^h = \frac{1}{1 + \frac{\zeta\pi_p}{1-\delta}}$$

and

$$z_s^l = \frac{1}{1 + \frac{\zeta\pi_p + \zeta\pi_s}{1-\delta}}.$$

Lemma 5.4. *Suppose that* $\lim_{\Delta\to 0} z_b > 0$. *Then*

$$\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\delta} = Q_p \equiv \frac{b+d}{d}\left[\frac{\phi_l - c_l}{c_h - \phi_l}\right], \tag{5.23}$$

$$\lim_{\Delta\to 0} \frac{\zeta\pi_s}{1-\delta} = Q_s \equiv \left[\frac{2c_h - (\phi_h + \phi_l)}{\phi_h - c_h}\right](1 + Q_p). \tag{5.24}$$

Proof. See Appendix C. □

Lemma 5.4 implies

$$\lim_{\Delta\to 0} z_s^h = \frac{1}{1 + Q_p} = \frac{c_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)}$$

and

$$\lim_{\Delta \to 0} z_s^l = \frac{1}{1+Q_s+Q_p} = \frac{\phi_h - c_h}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)}$$

which are independent of x_b. Thus, if $\lim_{\Delta \to 0} z_b > 0$, then

$$0 < \lim_{\Delta \to 0} z_s = \lim_{\Delta \to 0} z_s^h + z_s^l = \frac{\phi_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)}. \tag{5.25}$$

Under $A1 - A3$, one can easily verify that

$$\phi_h - \phi_l < 2(c_h - c_l)$$

from which

$$\frac{\phi_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)} < 2 \tag{5.26}$$

follows. □

Note that the right hand side of (5.22) is independent of z_b. $\lim_{\Delta \to 0} z_s$ is independent of how many buyers are in the pool, or whether or not buyers are in the short side, as long as a positive mass of buyers are in the pool.

This is yet another consequence of the fact that aggressive bargaining by the low quality sellers cause the failure of market clearing. As long as a positive mass of buyers is in the market, a low quality seller will be matched to a buyer almost surely within a short period of time, as $\Delta \to 0$. If a low quality seller knows she can meet another buyer very quickly after the present bargaining falls apart, she can assert a strong bargaining position, which leads to a small probability of reaching agreement, and eventually, prevents the market from clearing.

From (4.20), we know that z_b is a positive linear function of x_b. An increase of x_b affects z_s in two ways. As x_b increases, z_b increases, which increases the probability ζ of a seller meeting a buyer. One may conclude that if x_b increases, then z_s must decrease, since more sellers are matched away. This observation misses the second way that x_b affects z_s. As a seller faces a better chance of meeting a buyer, her long run average payoff increases, and she bargains more aggressively. As a result, π_s and π_p decrease as a linear function of ζ. In an equilibrium, the two effects of an increase in x_b are perfectly cancelled out so that z_s remains unaffected.

Our goal is to identify the conditions under which the market fails to clear:

$$\lim_{\Delta \to 0} z_b z_s > 0.$$

Thanks to Proposition 5.3, it suffices to completely characterize the conditions under which

$$\lim_{\Delta \to 0} z_b > 0.$$

Theorem 5.5.

$$\lim_{\Delta \to 0} z_b > 0$$

if and only if

$$\frac{\phi_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)} - (2 - x_b) > 0 \tag{5.27}$$

holds.

Combined with Lemma 5.2, we conclude that the buyer's equilibrium payoff must vanish, whenever the market fails to clear. Note that (5.27) can hold even if the mass of buyers is smaller ($x_b < 1$) than the mass of low quality sellers, where Akerlof (1970) predicts that the buyer should receive all equilibrium surplus from trading. We show otherwise, completely characterizing the condition under which the prediction from the static model is carried over to a dynamic model.

Observe that given other things, (5.27) will fail if the agents are very impatient so that b/d is large. For example, if $b/d = \infty$ and (5.27) fails, our model is essentially identical with the static model of Akerlof (1970), and the market clears in the sense that $\lim_{\Delta \to 0} z_b = 0$ when the buyers are on short side. The substance of Theorem 5.5 is to show that the intuition of Akerlof (1970) is carried over, as long as the agents are impatient in the sense that b/d is large.

The low quality seller can generate a large profit by agreeing on $p \in \Pi_p$ even if $\Delta > 0$ is small. However, trading at a high price from Π_p can be realized after possibly many rounds of matching and bargaining. If $b > 0$ is large so that (5.27) fails, then the seller is too impatient to exploit the future opportunity of trading at a high price, and is content with reaching an agreement quickly, which leads to $\lim_{\Delta \to 0} z_b = 0$, as Theorem 5.5 implies.

Proof. We state the proof for the necessity, while relegating the proof for the sufficiency to Appendix D. Suppose that

$$\lim_{\Delta \to 0} z_b > 0.$$

By Proposition 5.3, (5.22) holds. Since the matching is one to one,

$$z_b = x_b - 2 + z_s.$$

Substituting z_s by (5.22), we have

$$0 < \lim_{\Delta \to 0} z_b = \frac{\phi_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)} - (2 - x_b).$$

which implies (5.27) holds. □

The intuition for the sufficiency of (5.27) is as follows. Suppose b/d is small so that (5.27) holds. Indeed, if b/d is sufficiently close to zero, (5.27) always holds under $A1$-$A3$. Then the low type sellers will have an incentive to wait for the future gain from trading at a price in Π_p when a price in Π_s is drawn.

The more patient the agents are, the larger the gain from trading at a price drawn from Π_p becomes. The larger this gain becomes, the more aggressive the low quality seller becomes in bargaining, which leads to a lower probability of reaching an agreement. As b becomes small, π_s and π_p become so small at some point that they converge to zero at the rate of Δ. From (4.15), we have

$$\lim_{\Delta\to 0} W_b = 0.$$

Since $\lim_{\Delta\to 0} W_b + W_s^l = \phi_l - c_l$ holds due to Proposition 5.1, we have

$$\lim_{\Delta\to 0} W_s^l > 0.$$

One can rewrite (4.14) as

$$W_s^l = \zeta \left[\frac{\beta A(\pi_s)^2}{1-\beta\delta} + \frac{\beta\pi_p}{1-\beta\delta} \mathsf{E}[p - c_l - W_s^l | \Pi_p] \right]$$

Since π_s and π_p vanish at the rate of Δ, the term inside of the bracket is uniformly bounded. Since the left hand side is uniformly bounded away from 0, $\lim_{\Delta\to 0}\zeta > 0$ must hold.

5.3. **The analysis of the main model.** In the auxiliary model, $x_b > 0$ is an exogenous parameter. Instead, let us examine the main model in which buyers can enter the market freely, while a buyer has to pay vacancy cost $F > 0$ per unit of time, or ΔF per period while remaining in the pool, as in Mortensen and Pissarides (1994). Also, as in Mortensen and Pissarides (1994), if a buyer and a seller dissolve the existing long term relationship, then the buyer permanently exits from the economy, while the seller returns to the pool.

Due to the free entry condition,

$$W_b = 0 \tag{5.28}$$

must hold in any equilibrium. Due to the assumption of job destruction together with (5.28), we have

$$W_b = \beta\xi\mu_l\pi_s \mathsf{E}\left[W_b(p)|\Pi_s\right] + \beta\xi\pi_p\mathsf{E}\left[W_b(p)|\Pi_p\right] - (1-\beta)\Delta F = 0. \tag{5.29}$$

Since a buyer exits permanently after the existing long term relationship dissolves,

$$W_b(p) = \frac{1-\beta}{1-\beta\delta}(\phi^e(p) - p) \tag{5.30}$$

where $\phi^e(p)$ is the expected quality conditioned on reaching agreement at p.

Substituting (5.30) into (5.29) and noticing the properties of Π_s and Π_p, we obtain

$$\beta\xi\mu_l\pi_s \mathsf{E}\left[\frac{\phi_l - p}{1-\beta\delta}|\Pi_s\right] + \beta\xi\pi_p \mathsf{E}\left[\frac{\phi(\mu_l) - p}{1-\beta\delta}|\Pi_p\right] = \Delta F. \tag{5.31}$$

Substituting (4.12) and (4.13) with $W_b = 0$ into (5.31), we obtain

$$\frac{\beta A(\pi_s)^2\mu_l\xi}{1-\beta\delta} + \frac{\beta A(\pi_p)^2\xi}{1-\beta\delta} = \Delta F. \tag{5.32}$$

We claim that ξ is uniformly bounded away from zero. Suppose not, i.e., $\xi = O(\Delta^\alpha)$ for some $\alpha > 0$. Then (5.32) implies $(\pi_p)^2 = O(\Delta^{2-\alpha})$. From (4.14), W_s^l goes to infinity as Δ goes to zero, which is a contradiction. Next, using the same argument as in Lemma B.2, we can show that both π_s and π_p converge to zero at the same rate. Thus, in order to balance the rates of convergence between the left and right hand sides of (5.32), we must have $\pi_s = O(\Delta)$ and $\pi_p = O(\Delta)$. This implies, from the analysis of the auxiliary model, that $\lim_{\Delta\to 0} z_s > 0$ and $\lim_{\Delta\to 0} z_b > 0$ must hold. The market fails to clear.

6. Revelation of quality

To simplify notation, we have assumed so far that the true quality of the good is not revealed until the existing long term relationship is dissolved. In order to understand how the information revelation affects the equilibrium outcome, suppose that a buyer and a seller are in the long term relationship, who have agreed to deliver one unit of the good from the seller to the buyer at price p. After the good is delivered to the buyer, the true quality is revealed with probability $1-\lambda = 1-e^{-\Delta\theta}$ ($\theta > 0$). Based upon the available information about the good, if any, the buyer and the seller decide whether to continue the long term relationship or not. If both agents decide to continue the long term relationship, then the two agents remain in the same relationship with probability $\delta = e^{-\Delta d}$. Even if both agents choose to stay in the long term relationship, with probability $1-\delta$, the relationship is dissolved immediately, and the two agents return to their respective pools. If either agent decides to terminate the long term relationship, then the relationship is dissolved immediately and the two agents return to the respective pools. The rest of the rules of the game remain the same.

An important implication of the new information is that the buyer has an option to terminate the long term relationship, if he discovers the quality is low, and to continue the relationship, if the quality is high. While the new information allows the buyer to get rid of low quality goods, the ensuing analysis reveals that as long as the lemon's problem is severe, the results in the previous section are carried over.

Since the new information arrives in each period with a positive probability, however, we need to modify assumption $A3$ accordingly:

A3′. The lemons problem is severe in the sense that
$$\frac{\phi_h+\phi_l}{2}+\frac{1}{2}\frac{\theta}{b+d}\phi_h < c_h.$$

The first term of the left hand side is the average quality of the good when the good is purchased. After the good is purchased, the true quality is revealed with probability $1-e^{-\Delta\theta}$, while the agent discounts the future payoff at the rate of $e^{-\Delta b}$, and the long term relationship lasts with a probability of $e^{-\Delta d}$. After the true quality is revealed, only the high quality goods will be kept, which make up one half of the goods purchased by the buyer. The second term is the expected average discounted quality, conditioned on the event that the quality is revealed, and only the high quality good is kept.

Purchasing a good has an option value of observing the true quality, in addition to consuming the average quality. A tedious calculation shows that if price p is sufficiently high so that both high and low quality sellers agree to sell the good, the buyer accepts p when
$$\tilde{\phi}^e - p \geq W_b$$
where
$$\tilde{\phi}^e = \frac{\mu_l\phi_l + (1-\mu_l)\phi_h + \dfrac{\beta(1-\lambda)(1-\mu_l)}{1-\beta\delta}\phi_h}{1+\dfrac{\beta(1-\lambda)(1-\mu_l)}{1-\beta\delta}}.$$
Define
$$\Pi_p = [c_h + W_s^h, \tilde{\phi}^e - W_b] \qquad \text{and} \qquad \Pi_s = [c_l + W_s^l, \phi_l - W_b].$$
Then, we can calculate the value of each type of the agent conditioned on the event that he is in the pool:
$$W_s^h = \frac{\zeta\beta\pi_p \mathsf{E}(p-c_h-W_s^h \mid \Pi_p)}{1-\beta+\beta\lambda(1-\delta)},$$
$$W_s^l = \frac{\zeta\beta\pi_s \mathsf{E}(p-c_l-W_s^l \mid \Pi_s)}{1-\beta\lambda\delta} + \frac{\xi\beta\pi_p \mathsf{E}(p-c_l-W_s^l \mid \Pi_p)}{1-\beta\lambda\delta},$$
and
$$W_b = \frac{\xi\beta\mu_l\pi_s}{1-\beta\delta}\mathsf{E}(\phi_l-p-W_b \mid \Pi_s) + \frac{\zeta\beta\pi_p}{1-\beta\lambda\delta}\left(1+\frac{\beta(1-\lambda)(1-\delta)}{1-\beta\delta}\right)\mathsf{E}(\tilde{\phi}^e-p-W_b \mid \Pi_p).$$
These values can be rewritten in a form more convenient for the analysis if the price is drawn from a uniform distribution:
$$W_s^h = A\frac{\zeta\beta\pi_p^2}{1-\beta+\beta\lambda(1-\delta)},$$
$$W_s^l = A\frac{\zeta\beta\pi_s^2}{1-\beta\lambda\delta} + \frac{\zeta\beta\pi_p \mathsf{E}(p-c_l-W_s^l \mid \Pi_p)}{1-\beta\lambda\delta},$$

and

$$W_b = A\frac{\xi\beta\mu_l\pi_s^2}{1-\beta\delta} + A\frac{\xi\beta\pi_p^2}{1-\beta\lambda\delta}\left(1+\frac{\beta(1-\lambda)(1-\delta)}{1-\beta\delta}\right).$$

Along with the balance equations, we can solve for the equilibrium outcome $(z_b, z_s^h, z_s^l; W_b, W_s^h, W_s^l)$. We are interested in the case where

$$\lim_{\Delta\to 0} z_b > 0.$$

If

$$\lim_{\Delta\to 0} z_b > 0,$$

then $\lim_{\Delta\to 0} W_b = 0$ and $\lim_{\Delta\to 0} W_s^h = 0$ imply

$$\lim_{\Delta\to 0} \tilde{\phi}^e - c_h = 0.$$

Thus, we have

$$\lim_{\Delta\to 0} \mu_l = \frac{\left(1+\frac{\theta}{b+d}\right)\phi_h - c_h}{\left(1+\frac{\theta}{b+d}\right)\phi_h - \phi_l}.$$

We need to modify (5.27) accordingly:

$$\frac{\left(1+\frac{\theta}{b+d}\right)\phi_h - \phi_l}{c_h - c_l + \dfrac{b}{d}(\phi_l - c_l)} - (2 - x_b) > 0$$

which is a sufficient and necessary condition for

$$\lim_{\Delta\to 0} W_b = 0.$$

7. Concluding Remarks

This paper examines a dynamic matching model with adverse selection (Akerlof (1970) and Burdett and Wright (1998)) to see whether or not the market almost clears if search friction is small. We identify adverse selection as a fundamental source of the coexistence of unemployment and vacancy other than search friction and coordination failure caused by directed search.

Vacancy and unemployment are important objects of investigation in the labor market search models. A typical labor market search model (e.g., Mortensen and Pissarides (1994)) assumes a matching function $m(u, v)$ which specifies the rate at which unemployed workers (u) are matched to vacant positions (v). Indeed, Blanchard and Diamond (1989) pointed out that the matching function itself presumes the coexistence of a positive amount of unemployment (u) and a positive amount of vacancy (v). We have demonstrated that if the labor market is subject to adverse selection, then the equilibrium outcome can entail the coexistence of vacancy and unemployment, even in the limit as search friction vanishes.

We chose the random proposal model as a bargaining protocol mainly for the analytic convenience. The preliminary investigation reveals that the main conclusion of this paper is robust against the details of the bargaining protocols. In Cho and Matsui (2013a), for example, we demonstrate that the result is carried over to the model with a bargaining protocol in which the buyer makes the ultimatum offer in each period to the seller.

Appendix A. Proof of Proposition 5.1

Define $O(\Delta)$ as a function that vanishes at the rate of Δ:

$$\lim_{\Delta\to 0}\frac{O(\Delta)}{\Delta} < \infty.$$

Lemma A.1. $\lim_{\Delta\to 0}(\pi_p)^2 \le O(\Delta)$

Proof. The second term of the buyer's value function and $W_b < \infty$ imply the statement. □

Lemma A.2. $\lim_{\Delta\to 0}\frac{\zeta\pi_p}{1-\beta\delta} < \infty$.

Proof. Suppose $\lim_{\Delta\to 0}\frac{\zeta\pi_p}{1-\beta\delta} = \infty$. Since $\lim_{\Delta\to 0} W_s^l < \infty$,

$$\frac{\zeta\pi_p}{1-\beta\delta}\mathsf{E}(p - W_s^l - c_l \mid \Pi_p) < \infty.$$

Under the hypothesis of the proof,

$$\lim_{\Delta\to 0}\mathsf{E}(p - W_s^l - c_l \mid \Pi_p) = 0.$$

Since $\pi_p > 0$ and $\lim_{\Delta\to 0}\pi_p = 0$,

$$0 < \phi^e - W_b - c_l - W_s^l \to 0.$$

Recall

$$\phi_l < c_h.$$

Thus,

$$\phi_l - W_b < c_h + W_s^h$$

and the gap between the left and the right hand sides does not vanish as $\Delta \to 0$. Since $\pi_s > 0$,

$$c_l + W_s^l < \phi_l - W_b < c_h + W_s^h < \phi^e - W_b$$

while

$$\phi^e - W_b - c_l - W_s^l \to 0.$$

This is a contradiction. □

Based upon these two observations, we conclude that the high quality seller's equilibrium payoff vanishes as $\Delta \to 0$, which proves the first part of Proposition 5.1.

Lemma A.3. $\lim_{\Delta\to 0} W_s^h = 0$.

Proof. Apply Lemmata A.1 and A.2 to W_s^h. □

Since $\pi_s > 0$, an c_l seller and a buyer trades with a positive probability, which imposes an upper bound on $W_s^l + W_b$.

Lemma A.4. $W_s^l + W_b < \phi_l - c_l$.

Proof. A direct implication of $\pi_s > 0$. □

The next lemma shows that the low quality seller cannot be completely sorted out in a semi-pooling equilibrium, even in the limit as $\Delta \to 0$. As the pool contains a non-negligible portion of low quality sellers, the buyer needs to sort out the sellers, which is costly for the buyer and for the society as a whole, even if the friction vanishes. On the other hand, the low quality seller has an option to imitate the high quality seller, which provides significant bargaining power to a low quality seller when she is matched to a buyer.

Lemma A.5. $\lim_{\Delta\to 0}\mu_l > 0$.

Proof. Suppose $\lim_{\Delta\to 0}\mu_l = 0$. Then $\lim_{\Delta\to 0}\phi(\mu_l) = \phi_h$ holds. Thus, from (4.17), Lemmata A.3 and A.4 together with $W_s^l \geq 0$, we have

$$\lim_{\Delta\to 0}\pi_p = \lim_{\Delta\to 0} C[\phi_h - c_h - W_b - W_s^h] \geq C[(\phi_h - c_h) - (\phi_l - c_l)] > 0,$$

which contradicts with Lemma A.1. □

As in Lemma A.2, we can compute the rate at which $\zeta\pi_s$ vanishes.

Lemma A.6. $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} < \infty$.

Proof. Suppose $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} = \infty$. Then from Lemma A.2 and the balance equations of the sellers, $\lim_{\Delta\to 0}\mu_l = 0$ holds, which contradicts to Lemma A.5. □

The next lemma is the seller's counterpart of Lemma A.1.

Lemma A.7. $\lim_{\Delta\to 0}\pi_s \leq O(\Delta)$.

Proof. This statement is directly implied by Lemma A.5 and (4.15). □

A corollary of Lemma A.7 is that the sum of the long run average payoffs of a buyer and c_l seller converges to $\phi_l - c_l$, which proves the second part of Proposition 5.1.

Lemma A.8. $\lim_{\Delta\to 0} W_s^l + W_b = \phi_l - c_l$.

Proof. From Lemma A.7 together with (4.16), we have

$$\lim_{\Delta\to 0}\pi_s = \lim_{\Delta\to 0} C[(\phi_l - c_l) - (W_b + W_s^l)] = 0.$$

□

Appendix B. Proof of Lemma 5.2

From (4.18), (4.19) and (4.20), we know that in order to investigate the asymptotic properties of z_b and z_s, we need to understand the asymptotic properties of $\zeta\pi_p/(1-\delta)$ and $\zeta\pi_s/(1-\delta)$.

Lemma B.1. $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} > 0$

Proof. Suppose that $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} = 0$. From the balance equations of the sellers, we have

$$\frac{\mu_l}{1-\mu_l} = \frac{\frac{\pi_p\zeta}{1-\delta} + 1}{\frac{\pi_s\zeta}{1-\delta} + \frac{\pi_p\zeta}{1-\delta} + 1} \to 1$$

which implies that

$$\mu_l \to \frac{1}{2}.$$

Since the lemons problem is severe (assumption $A3$),

$$\phi(\mu_l) - c_h \to \frac{\phi_h + \phi_l}{2} - c_h < 0.$$

Recall that $W_s^h \to 0$. Since any equilibrium must be semi-pooling, $\pi_p > 0$. For a sufficiently small $\Delta > 0$, however,

$$0 < \phi(\mu_l) - W_b - W_s^h - c_h \leq \phi(\mu_l) - c_h \to \frac{\phi_h + \phi_l}{2} - c_h < 0$$

which is impossible. □

Lemma B.2.
$$0 < \lim_{\Delta \to 0} \frac{\pi_s}{\pi_p} < \infty.$$

Proof. Since we have
$$0 < \lim_{\Delta \to 0} \frac{\pi_s \zeta}{1-\delta} < \infty,$$
by way of Lemmata A.6 and B.1 , and
$$\lim_{\Delta \to 0} \frac{\pi_p \zeta}{1-\delta} < \infty,$$
by way of Lemma A.2,
$$\lim_{\Delta \to 0} \frac{\pi_p}{\pi_s} < \infty.$$
holds. To prove
$$\lim_{\Delta \to 0} \frac{\pi_p}{\pi_s} > 0$$
by way of contradiction, suppose that
$$\lim_{\Delta \to 0} \frac{\pi_p}{\pi_s} = 0.$$
Since
$$0 < \lim_{\Delta \to 0} \frac{\pi_s \zeta}{1-\delta} < \infty,$$
$$\lim_{\Delta \to 0} \frac{\pi_p}{\pi_s} = 0$$
implies
$$\lim_{\Delta \to 0} \frac{\pi_p \zeta}{1-\delta} = 0.$$
We claim that $\zeta \to 0$ as $\Delta \to 0$ under the hypothesis of the proof. If
$$\lim_{\Delta \to 0} \zeta > 0,$$
then $\pi_s = O(\Delta)$ and $\pi_p = O(\Delta)$. As a result,
$$\lim_{\Delta \to 0} W_s^l = \lim_{\Delta \to 0} W_b = 0,$$
which is impossible since
$$W_b + W_s^l \to \phi_l - c_l.$$
□

Lemma B.3. $\lim_{\Delta \to 0} \frac{\zeta \pi_p}{1-\beta\delta} > 0$

Proof. Note
$$\lim_{\Delta \to 0} \frac{\zeta \pi_p}{1-\beta\delta} = \lim_{\Delta \to 0} \frac{\zeta \pi_s}{1-\beta\delta} \frac{\pi_p}{\pi_s}.$$
The desired conclusion follows from Lemmata B.1 and B.2. □

Lemma B.4. $\lim_{\Delta \to 0} \mathsf{E}[p|\Pi_p] = c_h$.

Proof. Since $\lim_{\Delta \to 0} \pi_p = 0$, $\Pi_p = [c_h + W_s^h, \phi^e(p) - W_b]$ shrinks to a single point. Since $\lim_{\Delta \to 0} W_s^h = 0$, all points in Π_p converge to c_h, from which the conclusion follows. □

Lemma B.5. $\lim_{\Delta \to 0} W_s^l > 0$

Proof. Recall the equilibrium value function of W_s^l, and observe that the second term of the value function is strictly positive, even in the limit as $\Delta \to 0$. □

We are ready to prove Lemma 5.2. Note

$$\frac{W_s^l}{W_b} = \frac{A\zeta\pi_s^2 + \zeta\pi_p \mathsf{E}[p - c_l - W_s^l|\Pi_p]}{A\left(\mu_l\pi_s^2 + \pi_p^2\right)}.$$

Thus,

$$\frac{\mu_l W_s^l}{\zeta W_b} \propto \frac{\mu_l\zeta\pi_s^2 + \mu_l\zeta\pi_p\mathsf{E}(p - c_l - W_s^l|\Pi_p)}{\mu_l\zeta\pi_s^2 + \zeta\pi_p^2} = \frac{\mu_l\pi_s\frac{\pi_s}{\pi_p} + \mu_l\mathsf{E}(p - c_l - W_s^l|\Pi_p)}{\mu_l\pi_s\frac{\pi_s}{\pi_p} + \pi_p}. \tag{B.33}$$

The denominator converges to zero by way of Lemmata A.1, A.7, and B.2, while the numerator converges to a value greater than or equal to $\mu_l(c_h - \phi_l) > 0$ due to Lemma B.4 and $\lim_{\Delta\to 0} W_s^l \leq \phi_l - c_l$. Therefore, since $\lim_{\Delta\to 0}\mu_l W_s^l > 0$, $\zeta W_b \to 0$.

Appendix C. Proof of Lemma 5.4

Suppose $\lim_{\Delta\to 0} z_b > 0$. Then Lemma 5.2 implies $\lim_{\Delta\to 0} W_b = 0$, which in turn implies $\lim_{\Delta\to 0} W_s = \phi_l - c_l$ due to Proposition 5.1. We derive (5.23) from W_s^l by using the fact that the first term converges to zero, and Lemma B.4. As for (5.24), note that $\mu_l = z_s^l/z_s$. Taking the limit of this expression and equating it with $\lim_{\Delta\to 0}\mu_l = \frac{\phi_h - c_h}{\phi_h - \phi_l}$, we derive (5.24).

Appendix D. Proof of Theorem 5.5

We prove the sufficiency of (5.27) in multiple steps.

Proposition D.1. *Suppose that* $\lim_{\Delta\to 0} z_b = 0$. *Then,*

(1) $\lim_{\Delta\to 0} z_s = 2 - x_b$.
(2) $\frac{\pi_p}{1-\delta} \to \infty$ *and* $\frac{\pi_s}{1-\delta} \to \infty$ *as* $\Delta \to 0$.
(3) $\lim_{\Delta\to 0} W_b \geq 0$ *and the equality holds only if* (5.27) *is violated with equality.*
(4) (5.27) *is violated.*

Proof. Suppose $\lim_{\Delta\to 0} z_b = 0$.

(1) follows from the fact that $2 - z_s = x_b - z_b$.

(2) Note that $\zeta \to 0$ if and only if $z_b \to 0$. Lemma B.1 and Lemma B.3 imply that $\frac{\pi_p}{1-\delta} \to \infty$ and $\frac{\pi_s}{1-\delta} \to \infty$ as $\Delta \to 0$.

(3) To simplify notation, let us write

$$\begin{aligned}
\bar{\mu} &= \lim_{\Delta\to 0}\mu_l = \frac{\phi_h - c_h}{\phi_h - \phi_l} \\
\bar{Q}_s &= \lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\delta} \\
\bar{Q}_p &= \lim_{\Delta\to 0}\frac{\zeta\pi_p}{1-\delta}.
\end{aligned}$$

Under the assumption that $\zeta \to 0$, one can derive from the balance equations that

$$\frac{x_b}{2 - x_b} = \bar{\mu}\bar{Q}_s + \bar{Q}_p$$

and

$$\frac{\bar{\mu}}{1-\bar{\mu}} = \frac{\bar{Q}_p + 1}{\bar{Q}_s + \bar{Q}_p + 1}.$$

From the value function of c_l seller, one can show that

$$\lim_{\Delta\to 0} W_s^l = \frac{\frac{d}{b+d}\bar{Q}_p(c_h - c_l)}{1 + \frac{d}{b+d}\bar{Q}_p}.$$

Since

$$W_s^l + W_b \to \phi_l - c_l,$$

$\lim_{\Delta\to 0} W_b > 0$ if and only if

$$\frac{\frac{d}{b+d}\bar{Q}_p(c_h - c_l)}{1 + \frac{d}{b+d}\bar{Q}_p} < \phi_l - c_l.$$

We know that if $\bar{Q}_p = Q_p$, then

$$\frac{\frac{d}{b+d}\bar{Q}_p(c_h - c_l)}{1 + \frac{d}{b+d}\bar{Q}_p} = \phi_l - c_l.$$

Thus, $\lim_{\Delta\to 0} W_b > 0$ if and only if $\bar{Q}_p < Q_p$. One can show that $\bar{Q}_p$ solves

$$\bar{Q}_p + 1 = \left(1 + \frac{d}{d+b}\bar{Q}_p\right)\left(\frac{\phi_h - \phi_l}{c_h - c_l}\frac{1}{2 - x_b}\right),$$

where we use the balance equations, $\lim_{\Delta\to 0} W_s^l + W_b = \phi_l - c_l$, and

$$\bar{\mu}\phi_l + (1 - \bar{\mu})\phi_h = c_h + \lim_{\Delta\to 0} W_b.$$

Note that $\bar{Q}_p \leq Q_p$ if and only if (5.27) is violated, and the equality holds only if (5.27) is violated with an equality.

(4) follows from the last part of the proof of (3). □

References

Akerlof, G. A. (1970): “The Market for ”Lemons”: Quality Uncertainty and the Market Mechanism,” *Quarterly Journal of Economics*, 84(3), 488–500.

Ausubel, L. M., and R. J. Deneckere (1989): “Reputation in Bargaining and Durable Goods Monopoly,” *Econometrica*, 57(3), 511–531.

Azariadis, C. (1975): “Implicit Contracts and Underemployment Equilibria,” *Journal of Political Economy*, 83(6), 1183–1202.

Blanchard, O. J., and P. Diamond (1989): “The Beveridge Curve,” *Brookings Papers on Economic Activity*, 1989(1), 1–76.

Blouin, M. R., and R. Serrano (2001): “A Decentralized Market with Common Values Uncertainty: Non-Steady States,” *Review of Economic Studies*, 68, 323–346.

Burdett, K., S. Shi, and R. Wright (2001): “Pricing and Matching with Frictions,” *Journal of Political Economy*, 109(5), 1060–1085.

Burdett, K., and R. Wright (1998): “Two-Sided Search with Nontrasferable Utility,” *Review of Economic Dyanmics*, 1(1), 220–245.

Chang, B. (2012): “Adverse Selection and Liquidity Distortion in Decentrailed Markets,” Ph.D. thesis, Northwestern University.

Cho, I.-K., and A. Matsui (2012): “Competitive Equilibrium and Matching under Two Sided Incomplete Information,” University of Illinois and University of Tokyo.

——— (2013a): “A Dynamic Trading Model with Asymmetric Information and Take-It-or-Leave-It-Offer,” University of Illinois and University of Tokyo.

——— (2013b): “A Search Theoretic Foundation of Nash Bargaining Solution,” *Journal of Economic Theory*, 148, 1659–1688.

Diamond, P. A. (1971): “A Model of Price Adjustment,” *Journal of Economic Theory*, 3(2), 156–168.

Fuchs, W., and A. Skrzypacz (2013): “Costs and Benefits of Dynamic Trading in a Lemons Market,” *mimeo*.

Guerrieri, V., R. Shimer, and R. Wright (2010): “Adverse Selection in Competitive Search Equilibrium,” *Econometrica*, 78(6), 1823–1862.

Gul, F., and A. Postlewaite (1992): “Asymptotic Efficiency in Large Exchange Economies with Asymmetric Information,” *Econometrica*, 60(6), 1273–1292.

Harsanyi, J. C. (1956): “Approaches to the Bargaining Problem Before and After the Theory of Games: A Critical Discussion of Zeuthen’s, Hicks’, and Nash’s Theories,” *Econometrica*, 24(2), 144–157.

Keynes, J. M. (1936): *The General Theory of Employment, Interest and Money*. Macmillan.

Kim, K. (2012): “Endogenous Market Segmentation for Lemons,” *RAND Journal of Economics*, 43(3), 562–576.

Lagos, R. (2000): “An Alternative Approach to Search Frictions,” *Journal of Political Economy*, 108, 851–873.

Lauermann, S. (2013): “Dynamic Matching and Bargaining Games: A General Approach,” *American Economic Review*, 103(2), 663–689.

Lauermann, S., and A. Wolinsky (2013): “Search with Adverse Selection,” *mimeo*.

Matsui, A., and T. Shimizu (2005): “A Theory of Money and Market Places,” *International Economic Review*, 46.

Moreno, D., and J. Wooders (2010): “Decentralized Trade Mitigates the Lemons Problem,” *International Economic Review*, 51(2), 383–399.

Mortensen, D. T., and C. A. Pissarides (1994): “Job Creation and Job Destruction in the Theory of Unemployment,” *Review of Economic Studies*, 208, 397–415.

Rubinstein, A., and A. Wolinsky (1985): "Equilibrium in a Market with Sequential Bargaining," *Econometrica*, 53, 1133–1150.

Samuelson, W. (1984): "Bargaining under Asymmetric Information," *Econometrica*, 52(4), 995–1006.

Satterthwaite, M., and A. Shneyerov (2007): "Dynamic Matching, Two-Sided Incomplete Information and Participation Costs: Existence and Convergence to Perfect Equilibrium," *Econometrica*, 75(1), 155–200.

Shapiro, C., and J. E. Stiglitz (1984): "Equilibrium Unemployment as a Worker Discipline Device," *American Economic Review*, 74(3), 433–444.

Stiglitz, J. E., and A. Weiss (1981): "Credit Rationing in Markets with Imperfect Information," *American Economic Review*, 71(3), 393–410.

Vincent, D. R. (1989): "Bargaining with Common Values," *Journal of Economic Theory*, 48(1), 47–62.

Department of Economics, University of Illinois, 1206 S. 6th Street, Champaign, IL 61820 USA

E-mail address: inkoocho@uiuc.edu

URL: https://sites.google.com/site/inkoocho

Faculty of Economics, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

E-mail address: amatsui@e.u-tokyo.ac.jp

URL: http://www.e.u-tokyo.ac.jp/~amatsui