A Common Value Auction with Bid Solicitation*

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We study a first-price auction for a single good with common values. The novel feature is that the number of bidders is endogenous: the seller (auctioneer) knows the value and solicits bidders. Soliciting bidders is costly. The bidders privately observe noisy signals of the true value, as in the standard auction model. However, the number of solicited bidders is unobservable to them. The number of bidders that the seller solicits depends on the true value, giving rise to a solicitation effect*: Being solicited conveys information. The solicitation effect is a key difference to standard common value auctions. In contrast to standard auctions, bidders may pool on a common price, giving rise to atoms. We discuss information aggregation in the case of small bidder solicitation cost. We show that there is a type of equilibrium that aggregates information well when the most favorable signals are informative. However, there is also an equilibrium that fails to aggregate information.*

We study a first-price auction for a single good with common values. The novel feature is that the number of bidders is endogenous: the seller (auctioneer) knows the value and solicits bidders at a constant cost per sampled bidder. Thus, the number of bidders may vary across the different value states.

The bidders privately observe noisy signals of the true value, as in the standard auction model. However, the number of solicited bidders is unobservable to them. In equilibrium, bidders bid optimally given their signals, the behavior of others, and their beliefs concerning the sampling behavior of the seller. The seller chooses optimally how many bids to solicit given the bidders' behavior.

The number of solicited bidders in equilibrium depends on the true value, giving rise to a *solicitation effect:* The mere fact of being contacted by the seller and asked to submit a bid conveys information. The solicitation effect is a key difference to

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standard common value auctions. The relationship between the underlying value and the number of bidders is not always in the same direction. It is possible to have equilibria in which a seller of a high value good samples from bidders (solicitation is "good news," a solicitation *blessing*) and equilibria in which a seller of a low value good samples from bidders (solicitation is "bad news," a solicitation *curse*).

Our analysis has two objectives. First, we investigate the optimal solicitation strategy and the effects of the endogeneity of the number of bidders on their bidding strategies and on the auctioneer's revenue. Second, we study information aggregation—the relation between the expected winning bid (price) and the true value—when the cost of soliciting bidders is small.

The model has two states: one in which the value of the good is high for all bidders and another in which it is low for all bidders. The seller knows the true value and solicits bidders at random from a population of potential bidders. Solicited bidders obtain conditionally independent signals and participate in a first-price auction. We characterize symmetric equilibria in which all bidders use the same bidding strategies. If the seller solicits at least two bidders, these equilibria are monotone, in the sense that bidders who receive more favorable signals submit weakly higher bids. The seller's optimal solicitation strategy in each state has in its support either one or two adjacent integers.

We point out that the equilibrium bid distributions may exhibit atoms, that is, buyers may pool their bids. In fact, sometimes no equilibrium without atoms exists at all, that is, atoms may be unavoidable in equilibrium. This observation contrasts with the standard intuition that atoms induce bidders to overbid them slightly. This intuition fails here, since bidders may expect that on average more bids fall in the atom in the low state than in the high state, which could make overbidding unprofitable. One might think that a similar consideration could give rise to atoms in an ordinary common value auction, in which the number of bidders is constant across states. However, we show that this is not the case. In an ordinary common value auction, an atom may arise only at the bottom of the bid distribution and only if there is a positive probability that all buyers simultaneously observe the least favorable signal realization, that is, a mass point in the distribution of the signals at the bottom.

We also discuss the case of small bidder solicitation cost. We show that there are at most two kinds of distributions of the winning bid that may arise in equilibrium when the sampling cost is small: a nearly atomless distribution that is partially revealing and a distribution that is degenerate because bidders pool on a common price.

The partially revealing equilibrium qualitatively resembles the equilibrium outcome of an ordinary common value auction. It is partially revealing in the sense that the expected winning bid differs across the states. If the most favorable signal is very informative, the partially revealing case aggregates information well, in the sense that the winning bid is close to the true value.

The pooling equilibria fail to aggregate information poorly even when the sampling cost is small and the most favorable signals are very informative. Moreover, buyers may pool on a common price that is strictly below the ex ante expected value: The seller not only fails to receive a higher price when the good has a higher value, the seller may also not even receive a price equal to its expected value—despite the fact that a large number of bidders participate in the auction and bidders may obtain very informative signals.

We establish the existence of an equilibrium if the set of feasible bids is a fine grid. We show that a partially revealing equilibrium always exists. The pooling equilibrium exists under additional assumptions on the signal that we point out (signals must satisfy an increasing failure ratio property). We do not know whether a pooling equilibrium exists for all signal distributions.

Although we do not conduct the analysis with a particular application in mind, our analysis is relevant for some economically interesting scenarios. One such scenario is that of a potential borrower who possesses private information about the riskiness of the loan and applies to a number of lenders. The lenders obtain noisy signals and offer terms. They are aware that the borrower might be applying to other lenders but they may not know how many others. The borrower is the counterpart of the seller in our model (it is selling its bond) and the lenders are the bidders. Broecker (1990) and Riordan (1993) have modeled this situation as an ordinary common value auction. Accounting for endogenous (and unobservable) solicitation of terms by the borrower is natural in this environment and might produce new insights into the limits of competition in efficiently aggregating information in this setting.

Our work contributes to three strands of literature. First, for a common values auction environment, Wilson (1977) and Milgrom (1979) derived conditions on the informativeness of the signals under which the price aggregates information when the number of bidders becomes large. In their environment, the known number of bidders is independent of the state of nature and grows large exogenously. Kremer (2002) generalizes some of the earlier results. Second, Lauermann and Wolinsky (2011) study a *sequential search* environment with adverse selection, which is the sequential search counterpart of the common value auction environment. That study shows that information aggregation requires stronger conditions on the informativeness of signals in a sequential search environment than in the corresponding auction environment. Third, the present model can be interpreted as a simultaneous ("batch-")search model like Burdett-Judd (1983), with the added feature of adverse selection. In our Discussion in Section 8, we explain the relationships of our paper with this literature further.

Our results on the existence of equilibrium rely on a result by Athey (2001).

1 Model

Basics.—This is a single-good, common value, first-price auction environment with two underlying states, h and l. There are N potential bidders (buyers). The common values of the good for all potential bidders in the two states are v_l and v_h, with $0 \leq v_l < v_h$. The seller's cost is zero.

Nature draws a state $w \in \{l, h\}$ with prior probabilities $\rho_l > 0$ and $\rho_h > 0$, with $\rho_l + \rho_h = 1$. The seller learns the realization of the state w and invites n_w bidders, $n_w \leq N$. If $n_w < N$, the seller selects the invitees randomly with equal probability. We use $\mathbf{n}$ to denote the vector (n_l, n_h).

The seller incurs a solicitation cost $s > 0$ for each invited bidder. We assume that $N \geq \frac{v_h}{s}$. Therefore, N does not constrain the seller.

Each invited bidder observes a private signal $x \in [\underline{x}, \bar{x}]$ and submits a bid b from a set of feasible bids P_Δ. Conditional on the state, signals are independently and identically distributed according to a cumulative distribution G_w, $w \in \{h, l\}$. A bidder does not observe w nor how many other bidders are invited to bid.

The invited bidders bid simultaneously: The highest bid wins and ties are broken randomly with equal probabilities.

If the winning bid is p in state $w \in \{h, l\}$, then the payoffs are $v_w - p$ for the winning bidder and zero for all others. The seller's payoff is $p - n_w s$.

Further Details.—The set of feasible bids P_Δ may either be the full interval $[0, v_h]$ or a grid: For $\Delta > 0$, $P_\Delta = [0, v_l] \cup \{v_l + \Delta, v_l + 2\Delta, \cdots, v_h - \Delta, v_h\}$, so Δ measures the fineness of the grid. For $\Delta = 0$, it is the continuum $P_0 = [0, v_h]$. Much of the following analysis holds for both of the cases $\Delta > 0$ and $\Delta = 0$. We mention it explicitly when the discussion focuses on just one of these cases. Finally, even in the case of $\Delta > 0$ we leave the continuum of prices on $[0, v_l]$. This economizes on some largely irrelevant distinctions between the case in which the bottom equilibrium bid is v_l and the case in which it is $v_l - \Delta$.

The signal distributions G_h and G_l have identical supports, $[\underline{x}, \bar{x}] \subset \mathbb{R}$, and strictly positive densities g_l and g_h. The likelihood ratio $\frac{g_h(x)}{g_l(x)}$ is nondecreasing.

Thus, larger values of x indicate a higher likelihood of the higher value. We denote

$$\frac{g_h(\underline{x})}{g_l(\underline{x})} \equiv \lim_{x \to \underline{x}} \frac{g_h(x)}{g_l(x)} \quad \text{and} \quad \frac{g_h(\bar{x})}{g_l(\bar{x})} \equiv \lim_{x \to \bar{x}} \frac{g_h(x)}{g_l(x)}.$$

We assume that the signals are not trivial and boundedly informative,

$$0 < \frac{g_h(\underline{x})}{g_l(\underline{x})} < 1 < \frac{g_h(\bar{x})}{g_l(\bar{x})} < \infty.$$

By an abuse of notation, $G_w(A)$ also denotes the measure of the set of signals $A \subset [\underline{x}, \bar{x}]$. The assumption that the likelihood ratios are bounded implies that

$$G_h(A) = 0 \Leftrightarrow G_l(A) = 0,$$

that is, G_h and G_l are mutually absolutely continuous.

The prior likelihood ratio and the likelihood ratio at the most favorable signal $\bar{x}$ appear often in the analysis. We therefore dedicate to them special symbols,

$$\rho = \frac{\rho_h}{\rho_l} \qquad \text{and} \qquad \overline{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})}.$$

Expected Payoffs and Equilibrium.—The posterior probability of state $w \in \{l, h\}$ in the eyes of a bidder conditional on being solicited and receiving signal x is

$$\Pr[w|x] = \frac{\rho_w g_w(x) \frac{n_w}{N}}{\rho_l g_l(x) \frac{n_l}{N} + \rho_h g_h(x) \frac{n_h}{N}} = \frac{\rho_w g_w(x) n_w}{\rho_l g_l(x) n_l + \rho_h g_h(x) n_h}.$$

The terms $g_w(x)$ reflect the information contained in the signal, the terms $\frac{n_w}{N}$ reflect the information that is conveyed to the bidder by being invited, and the ρ_w reflect the prior information. Since the signals accrue only to bidders who were sampled, we do not need a separate piece of notation for the information that this bidder was sampled. Notice that N cancels out; for this reason the number of potential buyers does not play any role in the analysis. Rewriting Bayes' formula gives the posterior likelihood ratio,

$$\frac{\Pr[h|x]}{\Pr[l|x]} = \frac{\rho_h}{\rho_l} \frac{g_h(x)}{g_l(x)} \frac{n_h}{n_l}.$$

Three likelihood ratios determine the posterior: The prior likelihood ratio $\frac{\rho_h}{\rho_l}$, the signal likelihood ratio $\frac{g_h(x)}{g_l(x)}$, and the sampling likelihood ratio $\frac{n_h}{n_l}$.

We study pure and symmetric bidding strategies $\beta : [\underline{x}, \bar{x}] \to P_\Delta$ that are measurable. When there are n bidders who employ a bidding strategy β, the cumulative distribution of the *winning bid* in state w is

$$F_w\left(p|\beta,n\right)=G_w\left(\{x|\beta(x)\leq b\}\right)^n=\left[\int_{x|\beta(x)\leq p}g_w\left(x\right)dx\right]^n.$$

The expected winning bid with n bidders in state w is

$$\mathbb{E}\left[p|w;\,\beta,n\right]=\int_{\underline{x}}^{\bar{x}}pdF_w\left(p|\beta,n\right).$$

Let $\pi_w\left(b|\beta,n\right)$ be the probability of winning with bid b, given state w, bidding strategy β used by the other bidders, and n bidders. The expected payoff to a bidder who bids b, conditional on being solicited and observing the signal x, given the bidding strategy β and the solicitation strategy $\mathbf{n}=(n_l,n_h)$, is

$$U(b|x,\beta,\mathbf{n})=\frac{\rho_l g_l\left(x\right)\frac{n_l}{N}\pi_l\left(b|\beta,n_l\right)\left(v_l-b\right)+\rho_h g_h\left(x\right)\frac{n_h}{N}\pi_h\left(b|\beta,n_h\right)\left(v_h-b\right)}{\rho_l g_l\left(x\right)\frac{n_l}{N}+\rho_h g_h\left(x\right)\frac{n_h}{N}}. \quad (1)$$

It is sometimes instructive to write the expected payoffs as

$$\frac{\rho_l g_l\left(x\right)n_l}{\rho_l g_l\left(x\right)n_l+\rho_h g_h\left(x\right)n_h}\left[\pi_l(b|\beta,n_l)(v_l-b)+\frac{\rho_h g_h\left(x\right)n_h}{\rho_l g_l\left(x\right)n_l}\pi_h(b|\beta,n_h)\left(v_h-b\right)\right]. \quad (2)$$

Denote by $\Gamma_0\left(N,\mathbf{n},P_\Delta\right)$ the *bidding game* when the auctioneer is known to invite $\mathbf{n}=(n_l,n_h)$ bidders and the set of possible bids is P_Δ. A *bidding equilibrium* of $\Gamma_0\left(N,\mathbf{n},P_\Delta\right)$ is a strategy β such that, for all x, $b=\beta\left(x\right)$ maximizes $U(b|x,\beta,\mathbf{n})$ over P_Δ. The ordinary common value auction is a special case of the bidding game for $n_l=n_h$.

Denote by $\Gamma\left(s,P_\Delta\right)$ the overall game in which the potential number of bidders is $\left\lceil\frac{v_h}{s}\right\rceil$, the smallest natural number larger than $\frac{v_h}{s}$. A *pure equilibrium* of $\Gamma\left(s,P_\Delta\right)$ consists of a bidding strategy β and a solicitation strategy $\mathbf{n}=(n_l,n_h)$ such that (i) β is a bidding equilibrium of $\Gamma_0\left(\left\lceil\frac{v_h}{s}\right\rceil,\mathbf{n},P_\Delta\right)$ and (ii) the solicitation strategy is optimal for the seller,

$$n_w\in\underset{n\in\{1,2,....\}}{\arg\max}\,\mathbb{E}\left[p|w;\,\beta,n\right]-ns\qquad w\in\{l,h\}.$$

Since a pure equilibrium might not exist, we admit mixed solicitation strategies whereby the seller mixes over numbers of invited bidders. Let $\boldsymbol{\eta}=(\eta_l,\eta_h)$ denote a mixed solicitation strategy, where $\eta_w(n)$ is the probability with which $n=1,...,N$ bidders are invited in state w. In addition, let $\overline{n}_w(\eta_w)$ and $\overline{\pi}_w[b|\beta,\eta]$ denote the expected number of bidders and the weighted average probability of winning in state w.[1] These numbers are analogous to n_w and $\pi_w[b|\beta,n]$ in the deterministic solicitation case. To make the expressions less dense we omit here and later the argument of $\overline{n}_w(\eta_w)$ and write just $\overline{n}_w$ instead. The expected payoff to a bidder who bids b, conditional on being solicited and observing the signal x, given the common bidding strategy β and the solicitation strategy $\boldsymbol{\eta}=(\eta_l,\eta_h)$ is

$$U(b|x,\beta,\boldsymbol{\eta})=\frac{\rho_l g_l\left(x\right)\overline{n}_l\overline{\pi}_l[b|\beta,\eta_l](v_l-b)+\rho_h g_h\left(x\right)\overline{n}_h\overline{\pi}_h[b|\beta,\eta_h]\left(v_h-b\right)}{\rho_l g_l\left(x\right)\overline{n}_l+\rho_h g_h\left(x\right)\overline{n}_h}. \quad (3)$$

In a complete analogy to the above definitions, $\Gamma_0\left(N,\boldsymbol{\eta},P_\Delta\right)$ is the bidding game given $\boldsymbol{\eta}=(\eta_l,\eta_h)$ and $\Gamma\left(s,P_\Delta\right)$ is the full game. A bidding equilibrium of $\Gamma_0\left(N,\boldsymbol{\eta},P_\Delta\right)$ is a strategy β such that, for all x, $b=\beta\left(x\right)$ maximizes $U(b|x,\beta,\boldsymbol{\eta})$ over P_Δ. The strategy profile $(\beta,\boldsymbol{\eta})$ is an *equilibrium* of $\Gamma\left(s,P_\Delta\right)$ if (i) β is a bidding equilibrium of $\Gamma_0\left(N,\boldsymbol{\eta},P_\Delta\right)$ and (ii) the solicitation strategy is optimal,

$$\eta_w\left(n\right)>0\Rightarrow n\in\underset{n\in\{1,2,\dots.\}}{\arg\max}\,\mathbb{E}\left[p|w;\,\beta,n\right]-ns.$$

2 Bidding Equilibrium: Single Crossing, Bertrand, and Monotonicity of Bids

This section derives some properties of a bidding equilibrium strategy β. The main property is the monotonicity of the bidding equilibrium β when at least two bidders are invited in each of the states. If the likelihood ratio $\frac{g_h}{g_l}$ is strictly increasing everywhere, a bidding equilibrium β is necessarily nondecreasing. If the likelihood ratio is constant over some interval, all signals in this interval contain the same information. The bids need not be monotonic in a bidding equilibrium in which such signals are associated to different bids. Nevertheless, there is an equivalent bidding equilibrium that is monotonic and that is obtained by reordering the signals over such interval.

To state the result, we define the notion of an equivalent bidding equilibrium. A bidding equilibrium $\widetilde{\beta}$ is *equivalent* to a bidding equilibrium β if $\widetilde{\beta}\left(x\right)$ is an optimal bid for all signals and the distribution of bids in each state and, hence, the expected payoffs are identical, $G_w\left(\{x|\beta(x)\leq b\}\right)=G_w(\{x|\widetilde{\beta}(x)\leq b\})$ and $U(b|x,\widetilde{\beta},\boldsymbol{\eta})=U(b|x,\beta,\boldsymbol{\eta})$ for all b and $\boldsymbol{\eta}$. Let C be the set of signals at which the likelihood ratio $\frac{g_h(x)}{g_l(x)}$ is discontinuous. The set C may be empty and is at most countably large.

[1]With $\overline{n}_w(\eta_w)=\sum_{n=1}^{N}n\eta_w\left(n\right)$ and $\overline{\pi}_w\left(b|\beta,\eta_w\right)=\sum_{n=1}^{N}\eta_w\left(n\right)n\pi_w\left(b|\beta,n\right)/\overline{n}_w$.

Proposition 1 Bidding Equilibrium Characterization *Suppose $\boldsymbol{\eta}$ is such that $\eta_l(1) = \eta_h(1) = 0$, and β is a bidding equilibrium.*

1. *If $x' > x$, then $U(\beta(x')|x', \beta, \boldsymbol{\eta}) \geq U(\beta(x)|x, \beta, \boldsymbol{\eta})$. The inequality is strict if and only if $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$.*
2. *There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is nondecreasing on $[\underline{x}, \bar{x}] \backslash C$ and coincides with β over intervals over which $\frac{g_h}{g_l}$ is strictly increasing.*

The proof of Proposition 1 relies on two lemmas. The proofs of the statements use the fact that $\eta_l(1) = \eta_h(1) = 0$ implies that

$$\overline{\pi}_h[b|\beta, \eta_h] > 0 \Leftrightarrow \overline{\pi}_l[b|\beta, \eta_l] > 0,$$

since G_h and G_l are mutually absolutely continuous.

Lemma 1 Single-Crossing *Given any bidding strategy β, any solicitation strategy $\boldsymbol{\eta}$ and any bids $b' > b \geq v_l$.*

1. *If $\overline{\pi}_w[b'|\beta, \eta_w] > 0$ for some $w \in \{l, h\}$ then, for all $x' > x$,*

   $$U(b'|x, \beta, \boldsymbol{\eta}) \geq U(b|x, \beta, \boldsymbol{\eta}) \Rightarrow U(b'|x', \beta, \boldsymbol{\eta}) \geq U(b|x', \beta, \boldsymbol{\eta});$$

   If $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$, then $U(b'|x', \beta, \boldsymbol{\eta}) > U(b|x', \beta, \boldsymbol{\eta})$.
2. *If $\overline{\pi}_w[b'|\beta, \eta_w] = 0$ for some $w \in \{l, h\}$, then $\overline{\pi}_w[b|\beta, \eta_w] = 0$ and $U(b'|x, \beta, \boldsymbol{\eta}) = U(b|x, \beta, \boldsymbol{\eta}) = 0$ for all x.*

The following lemma collects a number of additional properties of a bidding equilibrium β. One of them is a straightforward Bertrand property: when the seller solicits two or more bids in both states, then $\beta(x) \geq v_l$, for all x.

Lemma 2 Bertrand *Suppose $\eta_l(1) = \eta_h(1) = 0$ and β is a bidding equilibrium.*

1. *$\overline{\pi}_w[\beta(x)|\beta, \eta_w] > 0$ if $\frac{g_h(x)}{g_l(x)} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$.*
2. *$\beta(x) \in [v_l, v_h)$ if $x > \underline{x}$,*
3. *$U(\beta(x')|x', \beta, \boldsymbol{\eta}) \geq U(\beta(x)|x, \beta, \boldsymbol{\eta})$ if $x' > x$. The inequality is strict if and only if $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$.*
4. *If $P_\Delta = [0, v_h]$, then $\beta(x) \in (v_l, v_h)$ for all $x > \underline{x}$ for which $\frac{g_h(x)}{g_l(x)} > \frac{g_h(\underline{x})}{g_l(\underline{x})}$.*

The proof of the lemma requires that the set of feasible bids is dense below v_l. If prices are from a grid even below v_l, it may be an equilibrium for all bidders to bid below v_l (just as there are non-zero price equilibria in the usual Bertrand pricing game with a price grid). To avoid such equilibria, the set of feasible bids is not discretized below v_l.

Proof of Proposition 1:

Part (1): Proved by Lemma 2.

Part (2)**:** Suppose that $\frac{g_h(x')}{g_l(x')} > \frac{g_h(x)}{g_l(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta(x') < \beta(x)$. Since β is a bidding equilibrium, $U(\beta(x)|x, \beta, \boldsymbol{\eta}) \geq U(\beta(x')|x, \beta, \boldsymbol{\eta})$. By Lemma 2, $\overline{\pi}_w[\beta(x')|\beta, \eta_w] > 0$ and $\beta(x') \geq v_l$. Therefore, by Lemma 1, $U(\beta(x)|x', \beta, \boldsymbol{\eta}) > U(\beta(x')|x', \beta, \boldsymbol{\eta})$, contradicting the optimality of $\beta(x')$ for x'. Thus, the supposition $\beta(x') < \beta(x)$ is false. Hence, $\beta(x') \geq \beta(x)$.

Next, suppose that $\frac{g_h(x')}{g_l(x')} = \frac{g_h(x)}{g_l(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta(x') < \beta(x)$. Then there is some interval containing x and x' over which $\frac{g_h(x)}{g_l(x)}$ is constant. Let $[x_-, x_+]$ be the closure of this interval. By the argument from before, $\beta(x'') \leq \beta(x)$ whenever $x'' < x_- < x$ and $\beta(x) \leq \beta(x''')$ whenever $x < x_+ < x'''$. Define $\widetilde{\beta}_1(x) = \beta(x)$ if $x \notin (x_-, x_+]$. If $x \in (x_-, x_+)$ define

$$\widetilde{\beta}_1(x) = \inf \{b : G_h(x) \leq G_h(\{t|\beta(t) \leq b\})\}.$$

If $\frac{g_h(x)}{g_l(x)}$ is continuous at x_+, we set $\tilde{\beta}_1(x_+) = \lim_{\varepsilon \to 0} \beta_\infty(x_+ - \varepsilon)$; otherwise, $\tilde{\beta}_1(x_+) = \beta(x_+)$. With this definition,

$$G_h(\{x|\widetilde{\beta}_1(x) \leq b\}) = G_h(\{x|\beta(x) \leq b\}),$$

for all b, that is, the distribution of bids induced by $\widetilde{\beta}_1$ is equal to the distribution of bids induced by β in state h. The induced distribution of bids is also the same in state l because $\beta = \widetilde{\beta}_1$ outside (x_-, x_+) and because the distributions G_l and G_h conditional on $x \in (x_-, x_+)$ are identical.

Because of the identity of the distribution of the bids in both states, the payoffs from all bids are identical. By construction, $\widetilde{\beta}_1$ is nondecreasing over (x_-, x_+) and over $(x_-, x_+]$ if $x_+ \notin C$ (the set of discontinuities of $\frac{g_h(x)}{g_l(x)}$). Bidding $\tilde{\beta}_1(x)$ is optimal for all $x \in (x_-, x_+)$ because every such bid must be in the support of β as well. For x_+, note that optimality is immediate if $x_+ \in C$, because $\tilde{\beta}_1(x_+) = \beta(x_+)$. If $x_+ \notin C$ and there is an atom at $b = \beta(x_+)$, then bidding b is optimal by continuity of payoffs in signals. If $x_+ \notin C$ and there is no atom at b, optimality follows from the fact that the payoffs $U(\beta(x_+ - \varepsilon)|\tilde{\beta}_1, x_+ - \varepsilon, \boldsymbol{\eta})$ converge to $U(\tilde{\beta}_1(x_+)|\tilde{\beta}_1, x_+, \boldsymbol{\eta})$.

Repeating this construction for all intervals over which $\frac{g_h(x)}{g_l(x)}$ is constant, we get a sequence of bidding strategies. Let $\tilde{\beta}$ be its pointwise limit on $(\underline{x}, \bar{x}]$ and $\tilde{\beta}(\underline{x}) = \lim_{\varepsilon \to 0} \beta(\underline{x} + \varepsilon)$. Then, $\tilde{\beta}$ is an equivalent bidding equilibrium. The bidding equilibrium is monotone on $[\underline{x}, \bar{x}] \setminus C$. ∎

Note that the proposition does not require that other bidders use a monotone (nondecreasing) bidding strategy. Therefore, every equilibrium is in monotone strategies.[2] In contrast, some existing single crossing conditions for auctions, such as the condition in Athey (2001), require monotonicity of the strategy of other bidders.

The proof of the single crossing condition avoids assuming monotonicity by using the two state assumption: The condition that $b \geq v_l$ implies that (i) the low state is unambiguously bad (profit is negative because the bid is higher than the value) and that (ii) the higher bid must be worse in the low state (because the increased probability of winning decreases profits in the low state). With more than two states, such a strong result may not hold and single crossing may require stronger assumptions (such as monotonicity) on the strategies of other bidders.

The following example of a bidding equilibrium strategy illustrates the significance of the assumption that at least two bidders are solicited in both states.

A Bidding Equilibrium that is not Weakly Increasing.—Suppose the distributions have common support $[0, 1]$, with $g_h(x) = 2x$ and $g_l(x) = 2 - 2x$, and suppose that $v_l > 0$. In this example, the signals $x = 1$ and $x = 0$ reveal the state.[3] Suppose that $n_h = 1$ and $n_l = 100$. No bidding equilibrium strategy is weakly increasing: To see why, note that $\pi_h[b|\beta, 1] = 1$ for all $b \geq 0$. Hence, $\beta(1) = 0$ in every bidding equilibrium. Now, suppose the bidding strategy were weakly increasing. Then, $\beta(x) = 0$ for all x—but this cannot be an equilibrium. Consider $x = 0$. The expected payoff from bidding $b = 0$ is $\frac{1}{100} v_l$ while the expected payoff from bidding $b' = \varepsilon$ is $v_l - \varepsilon$. Because $v_l > 0$, a deviation to b' is profitable for small ε.[4]

Intuitively, when the number of solicited bidders depends on the state, then signals inform bidders not only about the expected value but also about the number of competitors. If fewer buyers are solicited when $w = h$, a higher signal implies both a higher value and less competition. The example shows that this may lead a bidder with a higher signal to bid lower.

[2] The proof of the Proposition also applies if we allow other bidders to use mixed strategies. As a consequence, the Proposition implies that our restriction to pure strategies is without loss of generality.

[3] The example violates the assumption that likelihood ratios are bounded. This simplifies the argument but it is easily possible to change the example so that signals are boundedly informative while the equilibrium bids are still decreasing.

[4] In fact, one can show that β must be strictly decreasing on $[0, 1]$, using arguments analogous to the proof of Proposition 1.

3 Bidding Equilibrium: Atoms

One significant consequence of the endogenous solicitation of bidders is the emergence of atoms in the bidding equilibrium. In auctions with private values, a standard argument involving slight overbidding (or undercutting) precludes atoms in which bidders get positive payoffs. This argument does not apply directly to common value auctions, since overbidding the atom may have different consequences in different underlying states owing to possibly different frequency of bids that are tied in the atom in the different states. However, as is shown below, a somewhat more subtle argument still precludes atoms in a common value auction in which the number of bidders is independent of the state, $n_l = n_h = n$, except at the lowest equilibrium bid in the case of finite set of signals. However, when different numbers of bidders are solicited at different states, atoms are prevalent.

Example of an Atom in a Bidding Equilibrium.—Suppose that $v_l = 0$ and $v_h = 1$, with uniform prior $\rho_h = \rho_l = \frac{1}{2}$. Let $[\underline{x}, \bar{x}] = [0, 1]$, $g_h(x) = 0.8 + 0.4x$ and $g_l(x) = 1.2 - 0.4x$. Thus, $\frac{g_h(x)}{g_l(x)}$ is increasing as required. Pick some $\bar{b} \in [\frac{1}{3}, \frac{4}{10}]$.
Claim: Suppose $n_l = 6$ and $n_h = 2$. There is a bidding equilibrium in which

$$\beta(x) = \bar{b} \qquad \forall x \in [\underline{x}, \bar{x}].$$

Proof: The expected value conditional on x and winning with bid b is

$$\mathbb{E}[v|x, \text{ win at } b;\, \beta^k,\, \mathbf{n}] = \frac{1}{1 + \frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h}{n_l}\frac{\pi_h[b|\beta,n_h]}{\pi_l[b|\beta,n_l]}} v_l + \frac{\frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h}{n_l}\frac{\pi_h[b|\beta,n_h]}{\pi_l[b|\beta,n_l]}}{1 + \frac{\rho_h}{\rho_l}\frac{g_h(x)}{g_l(x)}\frac{n_h}{n_l}\frac{\pi_h[b|\beta,n_h]}{\pi_l[b|\beta,n_l]}} v_h.$$

At the atom, the assumption that ties are broken randomly implies that $\pi_h\left[\bar{b}|\beta, n_h\right] = \frac{1}{n_h} = \frac{1}{2}$ and $\pi_l\left[\bar{b}|\beta, n_l\right] = \frac{1}{n_l} = \frac{1}{6}$. Further, $\frac{\rho_h}{\rho_l} = 1$, $v_l = 0$, $v_h = 1$. Thus,

$$\mathbb{E}[v|x, \text{ win at } \bar{b};\, \beta^k,\, \mathbf{n}] = \frac{\frac{1}{1}\frac{g_h(x)}{g_l(x)}\frac{2}{6}\frac{\frac{1}{2}}{\frac{1}{6}}}{1 + \frac{1}{1}\frac{g_h(x)}{g_l(x)}\frac{2}{6}\frac{\frac{1}{2}}{\frac{1}{6}}} = \frac{\frac{g_h(x)}{g_l(x)}}{1 + \frac{g_h(x)}{g_l(x)}} \geq \frac{4}{10}.$$

The inequality is from $\frac{g_h(x)}{g_l(x)} \geq \frac{g_h(0)}{g_l(0)} = \frac{2}{3}$ for all x. Because $\bar{b} \leq \frac{4}{10}$, when bidding $\bar{b}$, almost all buyers expect strictly positive payoffs whereas undercutting $\bar{b}$ yields zero payoff. Consequently, for almost all signals (except possibly $x = \underline{x}$ if $\bar{b} = 0.4$), buyers strictly prefer bidding $\bar{b}$ to any $b < \bar{b}$.

There is also no incentive for any bidder to overbid $\bar{b}$. The expected value conditional on winning when overbidding $\bar{b}$ is

$$\mathbb{E}[v|x, \text{ win at } b > \bar{b};\, \beta^k,\, \mathbf{n}] = \frac{\frac{1}{1}\frac{g_h(x)}{g_l(x)}\frac{2}{6}\frac{1}{1}}{1+\frac{1}{1}\frac{g_h(x)}{g_l(x)}\frac{2}{6}\frac{1}{1}} \leq \frac{1}{3}.$$

The inequality is from $\frac{g_h(x)}{g_l(x)} \leq \frac{3}{2}$ for all x. Any bid above $\bar{b}$ is sure to win. Hence, because $\bar{b} \geq \frac{1}{3}$, bidding $b > \bar{b}$ yields strictly negative payoffs. However, when bidding $\bar{b}$, expected payoffs are positive. Therefore, for all signals, buyers strictly prefer bidding $\bar{b}$ to any $b > \bar{b}$. ∎

The key to the atom's immunity to deviations is the fact that $n_l > n_h$. Slightly overbidding the atom would result in a discontinuous increase in payoff in state h, but an even more significant decrease in state l. In other words, given the uniform tie-breaking rule, bidding in an atom provides insurance against winning too frequently ("hiding in the crowd") in the negative payoff state l.

If $n_l = 3n_h$ and n_h is sufficiently large, there exists no equilibrium in strictly increasing strategies. This is a consequence of a later result, Lemma 15. Consequently, atoms are "unavoidable" if the number of bidders depends on the state.

Finally, observe that bidding equilibria discussed here are not full equilibria. The seller's solicitation strategy is obviously not optimal. Optimal solicitation in a face of a single atom would be $n_l = n_h = 1$. We return, in Section 7.1, to the question of existence of a full equilibrium with an atom similar to the previous example.

Winning Probability at Atoms.—To continue the discussion of atoms, the following lemma derives an expression for the winning probability in the case of a tie. Define the generalized inverse of β by

$$x_-(p) = \inf\{x|\beta(x) \geq p\} \text{ and } x_+(p) = \sup\{x|\beta(x) \leq p\},$$

with $\underline{x} = \sup\emptyset$ and $\bar{x} = \inf\emptyset$.

Lemma 3 *Suppose β is nondecreasing. If $x_-(b) < x_+(b)$, then*

$$\pi_w(b|\beta, n) = \frac{G_w(x_+(b))^n - G_w(x_-(b))^n}{n\,(G_w(x_+(b)) - G_w(x_-(b)))}.$$

The lemma is proved in the Appendix.

Ordinary Common Value Auctions.—This subsection argues that a bidding equilibrium of a common value auction with $n_l = n_h = n$ essentially does not exhibit atoms. More precisely, a bidding equilibrium may have an atom only at the lowest possible bid and only if $\frac{g_h(x)}{g_l(x)}$ is constant at the bottom of the signal distributions (i.e., over some interval $[\underline{x}, \hat{x}]$) and only if the expected equilibrium payoff of those lowest bidders is zero. Thus, the bidding equilibrium cannot have an atom if the likelihood ratio $\frac{g_h(x)}{g_l(x)}$ is strictly increasing.

The absence of atoms is not trivial. Overbidding an atom may have different consequences in different underlying states owing to possibly different frequency of bids at the atom in the different states even though $n_l = n_h$. However, it turns out that this effect is not strong enough to deter deviations from atoms in this case in contrast with some cases of $n_l > n_h$ as shown in the example above.

Proposition 2 No Ties if $\mathbf{n}_l = \mathbf{n}_h$ *Suppose that $n_l = n_h = n \geq 2$ and β is a bidding equilibrium when $\Delta = 0$.*

1. *If $\frac{g_h(x)}{g_l(x)}$ is strictly increasing over $[\underline{x}, \bar{x}]$, then β is strictly increasing.*
2. *If $G_h(\{x|\beta(x) = p\}) > 0$ for some p (an atom), then $\beta(x) \geq p$ for almost all x, and $\beta(x') = p$ implies $U(p|x', \beta, \mathbf{n}) = 0$ and $\frac{g_h(x')}{g_l(x')} = \frac{g_h(\underline{x})}{g_l(\underline{x})}$.*

Thus, if there is an atom at some price p, then the price is at the bottom of the bid distribution, the expected payoffs of bidders in the atom is zero, and the likelihood ratio $\frac{g_h(x)}{g_l(x)}$ must be constant on some interval $[\underline{x}, \hat{x}]$.

4 Optimal Solicitation: Characterization

The seller's payoff when sampling n bidders who use bidding strategy β is $\mathbb{E}[p|w;\ \beta, n] - ns$. This expression is strictly concave whenever the bidding strategy is not constant. Consequently, there is either a unique optimal number of sampled bidders or the optimum is attained at two adjacent integers.

Lemma 4 Optimal Solicitation *Given any symmetric bidding function β, there is a number n^* such that*

$$\{n^*, n^*+1\} \supseteq \arg\max_n \mathbb{E}[p|w;\ \beta, n] - ns.$$

The lemma is an immediate consequence of the concavity of the first moment of the first-order statistic in the number of trials.

Proof of Lemma 4: The probability that the winning bid is below p is $\left(G_w\left(\beta^{-1}([0,p])\right)\right)^n$, where $\beta^{-1}([0,p]) = \{x : \beta(x) \in [0,p]\}$. By a standard argument, integration-by-parts for Riemann-Stieltjes integrals implies that the expected value of a random variable can be written as an integral of its (complementary) cumulative distribution function.[5] Therefore,

$$\mathbb{E}\left[p|w;\,\beta,n\right]=\int_0^{v_h}\left(1-\left(G_w\left(\beta^{-1}\left([0,t]\right)\right)\right)^n\right)dt.$$

The incremental benefit of soliciting one more bidder is therefore

$$\mathbb{E}\left[p|w;\,\beta,n+1\right]-\mathbb{E}\left[p|w;\,\beta,n\right]=\int_0^{v_h}\left(G_w\left(\beta^{-1}\left[0,t\right]\right)\right)^n\left(1-\left(G_w\left(\beta^{-1}\left[0,t\right]\right)\right)\right)dt. \tag{4}$$

The lemma is immediate whenever β is degenerate: If all buyers bid the same, the uniquely optimal number is $n^*=1$. If β is not degenerate, then inspection of the incremental benefit of soliciting one more bidder shows that it is strictly decreasing in n. Thus, the objective function is strictly concave, which implies the lemma. ■

Given the lemma, we can represent a mixed strategy η_w by $n_w\in\{1,...,N\}$ and $\gamma_w\in(0,1]$, where $\gamma_w=\eta_w(n_w)>0$ and $1-\gamma_w=\eta_w(n_w+1)\geq 0$. A solicitation strategy is pure if $\gamma_w=1$. Thus, from here on, when we talk about n_w in the context of a strategy η_w, we mean the bottom of the support of η_w. In fact, since our characterization results pertain to the case of small sampling costs and many bidders, they are not affected by whether or not the equilibrium strategies are actually pure or mixed. Mixed solicitation strategies matter only for the existence arguments.

Relative Solicitation Incentives: Inspection of the incremental benefit on an additional bidder provides some insights into how the incentive to solicit bidders depends on the seller's type. Suppose that β is nondecreasing. Then,

$$\mathbb{E}\left[p|w;\,\beta,n+1\right]-\mathbb{E}\left[p|w;\,\beta,n\right]=\int_0^{v_h}\left(G_w\left(x_+\left(p\right)\right)\right)^n\left(1-\left(G_w\left(x_+\left(p\right)\right)\right)\right)dp.$$

The incremental benefit depends on two terms, $\left(G_w\left(x_+\left(p\right)\right)\right)^n$—the probability that all n buyers bid below p—and $\left(1-\left(G_w\left(x_+\left(p\right)\right)\right)\right)$—the probability that the additional buyer bids higher. The monotone likelihood ratio property implies that $\left(G_l\left(x_+\left(p\right)\right)\right)^n\geq\left(G_h\left(x_+\left(p\right)\right)\right)^n$ while $1-\left(G_l\left(x_+\left(p\right)\right)\right)\leq 1-\left(G_h\left(x_+\left(p\right)\right)\right)$. Intuitively, if $w=l$, then the probability that the winning bid of the already sampled n sellers is high is smaller than if $w=h$. However, if $w=l$, the probability that the additional buyer bids high is low. The incentive to solicit bidders depends on the relative magnitudes of these two countervailing terms. As we demonstrate in examples, for given signal distribution, there can simultaneously be an equilibrium in

[5] With F the c.d.f. of some $y\in[0,b]$, $\int_0^b ydF(y)=[yF(y)]_0^b-\int_0^b Fdy=\int_0^b(1-F(y))\,dy$.

which the high type samples more bidders than the low type (*solicitation blessing*) and an equilibrium in which the low type samples more bidders (*solicitation curse*).

5 Large Numbers: Basic Results

This section obtains basic results that are needed for the subsequent characterization.

Lemma 5 Poisson Distribution *If* $\lim \left(G_l\left(x^k\right)\right)^{n_l^k} = q$ *for some sequence* $\left\{x^k, n^k\right\}$, *with* $\min\left\{n_l^k, n_h^k\right\} \to \infty$, *and* $\lim \frac{n_h^k}{n_l^k} \in (0,\infty)$, *then*

$$\lim_{k\to\infty}\Big(G_h(x^k)\Big)^{n_h^k} = q^{\frac{g_h(\bar{x})}{g_l(\bar{x})}\lim_{k\to\infty}\frac{n_h^k}{n_l^k}}.$$

The lemma is proved in the Appendix. An intuition for the Lemma is the following: For a given w and n_w^k, the number of signals above any cutoff x^k is binomially distributed, with n_w^k independent trials with success probabilities $1 - G_w(x^k)$. As is well-known, when the number of trials is large, the binomial distribution is well approximated by a Poisson distribution. Specifically, if $\lim n_w^k[1 - G_w(x^k)^{-1})] = \delta_w \in (0,\infty)$, then the number $m = \#(\text{signals weakly above } x^k)$ is Poisson distributed with parameter δ_w in the limit as $n_w^k \to \infty$. Therefore, $q = \lim G_w\left(x^k\right)^{n_w^k} = \Pr[m=0] = e^{-\delta_w}$.Now, $\lim \frac{n_h^k}{n_l^k} \in (0,\infty)$ implies that $\frac{\delta_h}{\delta_l} = \frac{g_h(\bar{x})}{g_l(\bar{x})}\lim \frac{n_h^k}{n_l^k}$. Thus,

$$\lim_{k\to\infty} G_h^{n_h^k}(x^k) = e^{-\delta_l\left(\frac{\delta_h}{\delta_l}\right)} = \left(\lim_{k\to\infty} G_l^{n_h^k}(x^k)\right)^{\left(\frac{\delta_h}{\delta_l}\right)} = q^{\frac{g_h(\bar{x})}{g_l(\bar{x})}\lim_{k\to\infty}\frac{n_h^k}{n_l^k}}.$$

Note this consequence of the Lemma: If β^k is nondecreasing, then

$$F_l(p|\beta^k, n_l^k) = G_l(x_+^k(p))^{n_h^k}.$$

Therefore, the Lemma implies that when $\lim F_l\left(p|\beta^k, n_l^k\right) = q$, then

$$\lim_{k\to\infty} F_h(p \mid \beta^k, n_h^k) = q^{\frac{g_h(\bar{x})}{g_l(\bar{x})}\lim_{k\to\infty}\frac{n_h^k}{n_l^k}}. \tag{5}$$

Lemma 6 Zero Profit in the Limit *For every* ε *there is an* $M(\varepsilon)$ *such that, if* $n_l > M(\varepsilon)$ *and* $n_h > M(\varepsilon)$*, then* $U(\beta(x)|x,\beta,n_l,n_h) < \varepsilon$ *for all* x *in every bidding equilibrium* β.

Consider a sequence of solicitation strategies $\boldsymbol{\eta}^k = (n_l^k, n_h^k)$ such that $\min\left\{n_h^k, n_l^k\right\} \to \infty$ and a corresponding sequence β^k of bidding equilibria. Lemma 6 implies that when $b^k \to b$ and $\lim \pi_l\left[b^k; \beta^k, n_l^k\right] > 0$ then

$$\limsup_{k\to\infty} \mathbb{E}[v|x, \text{ win at } b^k; \beta^k, \eta^k] \leq b. \tag{6}$$

In addition, for any sequence $\left\{x^k\right\}$ for which $\lim \beta^k\left(x^k\right) = b$, individual rationality requires that

$$\liminf_{k\to\infty} \mathbb{E}[v|x^k, \text{ win at } \beta^k(x^k); \beta^k, \eta^k] \geq b. \tag{7}$$

Therefore, $\lim \pi_l\left[\beta^k\left(x^k\right); \beta^k, n_l^k\right] > 0$ requires

$$\lim_{k\to\infty} \mathbb{E}[v|x^k, \text{ win at } \beta^k(x^k); \beta^k, \eta^k] = b. \tag{8}$$

Finally, we characterize the seller's solicitation strategy.

Lemma 7 Total Solicitation Costs *Consider a sequence $s^k \to 0$, and a sequence of bidding strategies β^k. Suppose that n_w^k is an optimal solicitation strategy given β^k in state w and $F_w\left(\cdot|\beta^k, n_w^k\right)$ converges pointwise. Then,*

$$\lim_{k\to\infty} n_w^k s^k = -\int_0^{v_h}\left(\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)\right)\ln\left(\lim_{k\to\infty} F_w\left(p|\beta^k, n_w^k\right)\right) dp.$$

The lemma allows us to characterize the total solicitation costs in the limit as a function of the distribution of the winning bid.

An intuition of the lemma is the following. Soliciting n^k bidders is optimal if the seller does not prefer to solicit αn^k bidders instead, where $0 < \alpha < 1$ or $1 < \alpha$. When soliciting αn^k bidders, the distribution of the winning bid changes to $F_w\left(p|\beta^k, \alpha n^k\right) = F_w\left(p|\beta^k, n^k\right)^\alpha$, because $F_w\left(p|\beta^k, \alpha n^k\right) = (G_w(\beta^{k-1}([0,p])))^{\alpha n^k}$. The expected payoff from soliciting αn^k bidders is

$$\int_0^{v_h}\left(1 - F_w\left(p|\beta^k, n^k\right)^\alpha\right) dp - \alpha\left(n^k s^k\right).$$

Ignoring integer constraints, soliciting n^k bidders is optimal if the derivative of the expected payoffs with respect to α is zero at $\alpha = 1$, that is, if

$$-\int_0^{v_h}\left(F_w\left(p|\beta^k, n^k\right)\right)\ln\left(F_w\left(p|\beta^k, n^k\right)\right) dp - n^k s^k = 0. \tag{9}$$

The lemma relies now on the observation that when $s^k \to 0$, either the number of solicited bidders is so large that the integer constraints can indeed be ignored or the number of optimally solicited bidders is bounded. In the latter case, the Lemma is shown to hold trivially because the distribution of the winning bids must become degenerate.

6 Characterization of Equilibria with Small Sampling Costs

We study the nature of the equilibrium bid distribution when the sampling cost is small. In particular, we inquire about the extent of information aggregation by the equilibrium winning bid, that is, whether the winning bid is near the true value when the sampling costs are small and many bidders may be sampled. The latter question relates to the work of Wilson (1977) and Milgrom (1981) in the context of ordinary common value auctions, without the solicitation element.

Overall, the analysis implies that there are at most two kinds of equilibrium outcomes when the sampling cost is negligible: A partially revealing outcome that qualitatively resembles the equilibrium outcome of an ordinary common value auction and a degenerate "pooling" outcome that is qualitatively different.

We consider a sequence of games $\Gamma\left(s^k, P_{\Delta^k}\right)$ indexed by k. The sampling cost s^k vanish to zero, $\lim s^k = 0$. We allow for both $\Delta^k = 0$ and $\Delta^k > 0$, with $\Delta^k \to 0$, that is, the grid becomes arbitrarily fine. Let β^k and $\boldsymbol{\eta}^k = (n_l^k, n_h^k)$ be equilibrium bidding and solicitation strategies for $\Gamma\left(s^k, P_{\Delta^k}\right)$. Recall that $F_w\left(p|\beta^k, \eta_w^k\right)$ denotes the cumulative distribution function of the winning bid. As a consequence of Helly's selection theorem, every sequence of cumulative distribution functions has a pointwise everywhere convergent subsequence.[6] We study the limit of such sequences, thinking of it as an approximation for $F_w\left(p|\beta^k, \eta_w^k\right)$ when s^k is small.

Define the function J as

$$J(r;\rho,\overline{g}) = \int_0^1 (x - \frac{1}{\overline{g}}) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1 + x\rho\overline{g}r)^2} dx. \tag{10}$$

Lemma 8 *There is a unique number $r^* = r^*(\rho, \overline{g}) \in (\frac{1}{\overline{g}}, \infty)$ s.t. $J(r^*; \rho, \overline{g}) = 0$.*

The proof is in the Appendix. The solution r^* plays an important role in the characterization of equilibrium and the significance of $J(r^*; \rho, \overline{g}) = 0$ will become clear later.

Given the parameters $\rho = \frac{\rho_h}{\rho_l} > 0$, $\overline{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})} > 1$, and $r^* = r^*(\rho, \overline{g})$, we define

$$\bar{p}\left(\rho, \bar{g}\right) = \frac{v_l + \rho\overline{g}r^* v_h}{1 + \rho\overline{g}r^*},$$

[6] This is immediate from the monotonicity of F_w; see Kolmogorov and Fomin (1970, p. 372).

as well as two functions,

$$\bar{F}_l\left(p|\rho,\overline{g}\right)=\begin{cases}1 & \text{if} \quad p\geq\bar{p},\\ \left(\frac{1}{\rho\overline{g}r^*}\frac{p-v_l}{v_h-p}\right)^{\frac{1}{\overline{g}r^*-1}} & \text{if} \quad v_l<p\leq\bar{p},\\ 0 & \text{if} \quad p\leq v_l,\end{cases} \tag{11}$$

and

$$\bar{F}_h\left(\cdot|\rho,\overline{g}\right)=\left(\bar{F}_l\left(\cdot|\rho,\overline{g}\right)\right)^{\overline{g}r^*}.$$

Since $\overline{g}r^*>1$ by the definition of r^*, $\bar{F}_h$ and $\bar{F}_l$ are cumulative distribution functions with support $[v_l,\bar{p}]$. Both functions are strictly increasing on $(v_l,\bar{p})$ and continuously differentiable.

The following proposition characterizes the set of possible equilibrium outcomes when solicitation costs are negligible.[7]

Proposition 3 Equilibrium Characterization *Consider a sequence of games* $\Gamma\left(s^k,P_{\Delta^k}\right)$*, where* $s^k>0$*,* $\Delta^k\geq 0$ *and* $\lim\left(s^k,\Delta^k\right)=(0,0)$*. Suppose that* $(\beta^k,\boldsymbol{\eta}^k)$ *is a corresponding sequence of equilibria such that* $\frac{n_h^k}{n_l^k}$ *and* $F_w\left(\cdot|\beta^k,\eta_w^k\right)$ *converge.*

1. *If* $\frac{g_h(\bar{x})}{g_l(\bar{x})}\lim\frac{n_h^k}{n_l^k}>1$ *and* $\min\left\{n_l^k,n_h^k\right\}\to\infty$ *, then* $\lim\frac{n_h^k}{n_l^k}=r^*\left(\rho,\overline{g}\right)$ *and*

$$\lim_{k\to\infty}F_w\left(p|\beta^k,\eta_w^k\right)=\bar{F}_w\left(p|\rho,\overline{g}\right)\qquad\forall p\in\left[0,v_h\right],\ w\in\{l,h\}.$$

2. *Otherwise, the distribution of the winning bid converges to a degenerate distribution with probability mass one on some number* $C\leq E_0\left[v\right]$*.*

An immediate implication of the proposition is that for sufficiently small (s,Δ), every equilibrium is close to either the partially separating or a pooling equilibrium in the following sense.

Corollary 1 *For every* $\varepsilon>0$ *there is a* $\kappa>0$ *so that whenever* $(s,\Delta)\leq(\kappa,\kappa)$ *and* $(\beta,\boldsymbol{\eta})$ *is an equilibrium of* $\Gamma\left(s,P_\Delta\right)$*, then either*

$$\left|F_w\left(p|\beta,\eta_w\right)-\bar{F}_w\left(p|\rho,\overline{g}\right)\right|\leq\varepsilon\qquad\forall p\in\left[0,v_h\right],\ w\in\{l,h\},$$

or there is some $C\leq E_0\left[v\right]$ *such that* $F_w\left(C-\varepsilon|\beta,\eta_w\right)\leq\varepsilon$ *and* $F_w\left(C+\varepsilon|\beta,\eta_w\right)\geq 1-\varepsilon$ *for* $w\in\{l,h\}$*.*

[7] Recall that by Lemma 4 we represent a mixed equilibrium strategy η_w by n_w and γ_w, where $\gamma_w=\eta_w\left(n_w\right)$ and $1-\gamma_w=\eta_w\left(n_w+1\right)$.

Of course, any game $\Gamma(s, P_\Delta)$ may have multiple equilibria. In Section 7 we show that multiple equilibria do, in fact, exist under some assumptions, constructing one equilibrium that is close to the separating outcome and another equilibrium that is close to the pooling outcome.

The proposition identifies $\frac{g_h(\bar{x})}{g_l(\bar{x})} \lim \frac{n_h^k}{n_l^k}$ as a key magnitude in the nature of the equilibrium distribution of the winning bid. When this expression is larger than one, the limiting equilibrium bid distribution is atomless and the ratio of the number of solicited bidders and the distribution of the winning bid are unique. In this case, the equilibrium is partially revealing in the sense that the distribution differs across the states. But even in the limit, the expected winning bid does not coincide with the true value. When $\frac{g_h(\bar{x})}{g_l(\bar{x})} \lim \frac{n_h^k}{n_l^k} \leq 1$, the limiting equilibrium bid distribution becomes degenerate.

The magnitude $\frac{g_h(\bar{x})}{g_l(\bar{x})} \frac{n_h^k}{n_l^k}$ is meaningful. If it is smaller (larger) than 1, a sampled bidder who observes the highest possible signal is more pessimistic (optimistic) about h than she would be based on the prior information alone.

Given some solicited bidder, let $x_{(1)}$ denote the highest signal among its competitors. $\mathbb{E}\left[v|x,\, x_{(1)} \leq x,\, \beta^k,\, \boldsymbol{\eta}^k\right]$ is the expected value conditional on being sampled, observing signal x, and having the highest signal. If $\lim \frac{g_h(\bar{x})}{g_l(\bar{x})} \frac{n_h^k}{n_l^k} > 1$, then $\mathbb{E}\left[v|x,\, x_{(1)} \leq x;\, \beta^k,\, \boldsymbol{\eta}^k\right]$ is strictly increasing in the signal x, for sufficiently large x and k. Conversely, if $\lim \frac{g_h(\bar{x})}{g_l(\bar{x})} \frac{n_h^k}{n_l^k} < 1$, then $\mathbb{E}\left[v|x,\, x_{(1)} \leq x;\, \beta^k,\, \boldsymbol{\eta}^k\right]$ is strictly decreasing in x for sufficiently large x and k. These two facts are critical in determining the shape of the bidding distribution. In the first case, equilibrium bids are shown to be essentially strictly increasing in the signals in the relevant range. In the second case, an equilibrium in strictly increasing strategies cannot exist and equilibria must exhibit atoms.

6.1 Proof of Proposition 3.1: Partially Revealing Equilibria

We consider a sequence of equilibria $(\beta^k, \boldsymbol{\eta}^k)$ that satisfy the hypothesis of Proposition 3.1, that is, the distributions F_w and the ratio $\frac{n_h^k}{n_l^k}$ converge, and, in addition, $\min\left\{n_l^k, n_h^k\right\} \to \infty$ and $\frac{g_h(\bar{x})}{g_l(\bar{x})} \lim \frac{n_h^k}{n_l^k} > 1$. We abbreviate the limit ratio frequently as

$$\hat{r} = \lim_{k \to \infty} \frac{n_h^k}{n_l^k}.$$

By Proposition 1, for every β^k there is an equivalent bidding strategy which is nondecreasing, except possibly at the countable set of discontinuity points of $\frac{g_h(x)}{g_l(x)}$. In the following, we assume that β^k is nondecreasing everywhere. As in the proof of

Proposition 2, the proof can easily be extended to account for non-monotonicities at discontinuity points.

Recall from Proposition 4 that the support of an equilibrium solicitation strategy η_w contains at most two adjacent numbers $\{n_w, n_w+1\}$. Since we are dealing with limit values, it is without loss of generality to conduct the argument in terms of singleton solicitation strategies, so that η_w and n_w are identified. This simplifies the notation. Everything we do below remains essentially unchanged if we allow non-singleton solicitation strategies, except that we sometimes have to include γ_w's in the expressions in a way that does not change anything of importance in the analysis.

First, we show that there are no atoms in the equilibrium distribution of the winning bid in the limit.

Lemma 9 No Atoms in Limit *For any sequence* $\{x^k\}$ *and* $w \in \{l,h\}$,

$$\lim_{k\to\infty}(\pi_w[\beta^k\left(x^k\right)|\beta^k,\eta^k_w] - G_w(x^k_+(\beta^k(x^k)))^{n^k_w-1}) = 0.$$

The proof of the lemma is in the Appendix. For the subsequent lemmas, recall

$$F_w(p|\beta^k, n^k_w) = G_w(x^k_+(p))^{n^k_w}. \tag{12}$$

It follows that $\lim F_h\left(p|\beta^k, n^k_h\right) \in (0,1)$ implies that $x^k_+(p) \to \bar{x}$. For the following lemma, recall that $x_{(1)}$ denotes the highest signal among the competitors of a fixed bidder.

Lemma 10 *For every price for which* $\lim F_h\left(p|\beta^k, n^k_h\right) \in (0,1)$,

$$p = \lim_{k\to\infty}\mathbb{E}[v|x^k_+(p),\, x_{(1)} \le x^k_+(p),\, \beta^k,\, \boldsymbol{\eta}^k].$$

Proof: Fix a price p for which $\lim F_h\left(p|\beta^k, n^k_h\right) \in (0,1)$. Let $b^k_+ = \lim_{\varepsilon\to 0}\beta^k\left(x^k_+(p)+\varepsilon\right)$ and $b^k_- = \lim_{\varepsilon\to 0}\beta^k\left(x^k_+(p)-\varepsilon\right)$. We can choose sequences of signals y^k_-, y^k_+ such that $y^k_- \le x^k_+(p) \le y^k_+$, $\beta^k\left(y^k_-\right) \le b^k_-$, $b^k_+ \le \beta^k\left(y^k_+\right)$, and

$$\lim_{k\to\infty} G_w(y^k_-)^{n^k_w-1} = \lim_{k\to\infty} G_w(x^k_+(p))^{n^k_w-1} = \lim_{k\to\infty} G_w(y^k_+)^{n^k_w-1},$$

for $w \in \{l,h\}$. Therefore, after dropping β^k, $\boldsymbol{\eta}^k$ from the arguments of the expected values,

$$\lim_{k\to\infty}\mathbb{E}\left[v|y^k_-,\, x_{(1)} \le y^k_-\right] = \lim_{k\to\infty}\mathbb{E}\left[v|x^k_+(p),\, x_{(1)} \le x^k_+(p)\right] = \lim_{k\to\infty}\mathbb{E}\left[v|y^k_+,\, x_{(1)} \le y^k_+\right], \tag{13}$$

because $\lim y_-^k = \lim y_+^k = \bar{x}$ and because $\lim G_w \left(x_+^k (p)\right)^{n_w^k - 1} > 0$ by (12) and the hypothesis of the Lemma. From Lemma 9,

$$\begin{aligned} \lim_{k\to\infty} \pi_w \left[\beta^k \left(y_-^k\right) | \beta^k, \eta_w^k\right] &= \lim_{k\to\infty} G_w \left(y_-^k\right)^{n_w^k - 1}, & (14)\\ \lim_{k\to\infty} \pi_w \left[\beta^k \left(y_+^k\right) | \beta^k, \eta_w^k\right] &= \lim_{k\to\infty} G_w \left(y_+^k\right)^{n_w^k - 1}, & (15) \end{aligned}$$

for $w \in \{l, h\}$. From (14) and (15),

$$\begin{aligned} \lim_{k\to\infty} \mathbb{E}\left[v | y_-^k, \text{ win at } \beta^k \left(y_-^k\right)\right] &= \lim_{k\to\infty} \mathbb{E}\left[v | y_-^k,\, x_{(1)} \leq y_-^k\right], & (16)\\ \lim_{k\to\infty} \mathbb{E}\left[v | y_+^k, \text{ win at } \beta^k \left(y_+^k\right)\right] &= \lim_{k\to\infty} \mathbb{E}\left[v | y_+^k,\, x_{(1)} \leq y_+^k\right]. & (17) \end{aligned}$$

Combining (16) and (17) with the zero profit condition,

$$\begin{aligned} \lim_{k\to\infty} \beta^k \left(y_-^k\right) &= \lim_{k\to\infty} \mathbb{E}\left[v | y_-^k,\, x_{(1)} \leq y_-^k\right], & (18)\\ \lim_{k\to\infty} \beta^k \left(y_+^k\right) &= \lim_{k\to\infty} \mathbb{E}\left[v | y_-^k,\, x_{(1)} \leq y_+^k\right]. & (19) \end{aligned}$$

Hence, $\beta^k \left(y_-^k\right) \leq b_-^k \leq p \leq b_+^k \leq \beta^k \left(y_+^k\right)$ for all k, (18), (19), and (13) imply

$$p = \lim_{k\to\infty} \mathbb{E}\left[v | x_+^k (p),\, x_{(1)} \leq x_+^k (p)\right],$$

as claimed. ∎

Given parameters $\rho > 0$, $\overline{g} > 1$, we define

$$\phi_l (p | \rho, \overline{g}, r) = \begin{cases} 1 & \text{if} \quad p \geq \frac{v_l + \rho \overline{g} r v_h}{1 + \rho \overline{g} r}, \\ \left(\frac{1}{\rho \overline{g} r} \frac{p - v_l}{v_h - p}\right)^{\frac{1}{\overline{g} r - 1}} & \text{if} \quad v_l < p \leq \frac{v_l + \rho \overline{g} r v_h}{1 + \rho \overline{g} r}, \\ 0 & \text{if} \quad p \leq v_l, \end{cases} \tag{20}$$

and

$$\phi_h (\cdot | \rho, \overline{g}, r) = \left(\phi_l (\cdot | \rho, \overline{g}, r)\right)^{\overline{g} r}.$$

The function $\bar{F}_w (\cdot | \rho, \overline{g})$ defined before is a special case of $\phi_w (\cdot | \rho, \overline{g}, r)$ with $r = r^*$. Recall that $\hat{r} = \lim \frac{n_h^k}{n_l^k}$.

Lemma 11 $\lim F_w \left(p | \beta^k, n_w^k\right) = \phi_w (p | \rho, \overline{g}, \hat{r})$ *for* $w \in \{l, h\}$ *and* $p \in [0, v_h]$.

Proof: Choose p such that $\lim F_w \left(p | \beta^k, n_w^k\right) \in (0, 1)$. From the previous Lemma,

$p = \lim \mathbb{E}\left[v|x_+^k,\, x_{(1)} \le x_+^k\,(p)\right]$. Since $x_+^k(p) \to \bar{x}$, this is equivalent to

$$\frac{\rho_h}{\rho_l}\frac{g_h\left(\bar{x}\right)}{g_l\left(\bar{x}\right)}\lim_{k\to\infty}\frac{n_h^k}{n_l^k}\frac{G_h\left(x_+^k(p)\right)^{n_h^k}}{G_l\left(x_+^k(p)\right)^{n_l^k}} = \frac{p-v_l}{v_h-p}, \tag{21}$$

observing that $\lim F_w\left(p|\beta^k, n_w^k\right) > 0$ implies $p < v_h$ .By Lemma 5 and (5),

$$\lim F_h\left(p|\beta^k, n_l^k\right) = \left[\lim F_l\left(p|\beta^k, n_h^k\right)\right]^{\frac{g_h(\bar{x})}{g_l(\bar{x})}\lim\left(\frac{n_h^k}{n_l^k}\right)}. \tag{22}$$

With (12), $G_w\left(x_+^k(p)\right)^{n_w^k} = F_w\left(p|\beta^k, n_w^k\right)$, equations (21) and (22) imply

$$\lim_{k\to\infty} F_l\left(p|\beta^k, n_l^k\right) = \left(\frac{1}{\rho\overline{g}\hat{r}}\frac{p-v_l}{v_h-p}\right)^{\frac{1}{\overline{g}\hat{r}-1}}.$$

with $\hat{r} = \lim\frac{n_h^k}{n_l^k}$, $\rho = \frac{\rho_h}{\rho_l}$ and $\overline{g} = \frac{g_h(\bar{x})}{g_l(\bar{x})}$. Hence, $\lim F_l\left(p|\beta^k, n_l^k\right) = \phi_l\left(p|\rho,\overline{g},\hat{r}\right)$ if $\lim F_l\left(p|\beta^k, n_l^k\right) \in (0,1)$. Using Lemma 5 again implies $\lim F_h\left(p|\beta^k, n_h^k\right) = \phi_h\left(p|\rho,\overline{g},\hat{r}\right)$. Finally, the monotonicity of F_w and the definition of ϕ_w imply that $\lim F_w\left(p|\beta^k, n_w^k\right) \in \{0,1\} \Leftrightarrow \phi_w\left(p|\rho,\overline{g},\hat{r}\right) \in \{0,1\}$. ∎

The previous lemma determines the distribution of the winning bid as a function of the limit ratio $\hat{r} = \lim\frac{n_h^k}{n_l^k}$. We now use the seller's optimality condition to determine the limit ratio. Lemma 7 and Lemma 11 imply that

$$\lim_{k\to\infty} n_h^k s^k = -\int_0^{v_h}\left(\phi_h\left(p|\rho,\overline{g},\hat{r}\right)\right)\ln\left(\phi_h\left(p|\rho,\overline{g},\hat{r}\right)\right)dp.$$

Because$\frac{1}{\hat{r}}\lim\left(n_h^k s^k\right) = \lim\left(n_l^k s^k\right)$, Lemma 7 for $w = l$ requires

$$\frac{1}{\hat{r}}\int_0^{v_h}\left(\phi_h\left(p|\rho,\overline{g},\hat{r}\right)\right)\ln\left(\phi_h\left(p|\rho,\overline{g},\hat{r}\right)\right)dp = \int_0^{v_h}\left(\phi_l\left(p|\rho,\overline{g},\hat{r}\right)\right)\ln\left(\phi_l\left(p|\rho,\overline{g},\hat{r}\right)\right)dp.$$

We show in the Appendix that the above equation holds if and only if $J(\hat{r};\rho,\overline{g}) = 0$.

Lemma 12 *$J\left(\hat{r};\rho,\overline{g}\right) = 0$ and $\hat{r} < \infty$.*

Proof of Proposition 3.1: Recall from Lemma 8 that $r^* = r^*(\rho,\overline{g})$ is the unique solution from $(\frac{1}{\overline{g}},\infty)$ of the equation $J(r;\rho,\overline{g}) = 0$. This, together with Lemma 12, implies that, for any sequence of equilibria such that $\overline{g}\lim\frac{n_h^k}{n_l^k} > 1$, it must be that $\lim\frac{n_h^k}{n_l^k} = r^*$. Therefore, Lemma 11 implies that, for any such sequence,

$$\lim_{k\to\infty} F_w(p|\beta^k, n_w^k) = \phi_w\left(p|\rho,\overline{g},r^*\right), \qquad \forall p \in [0,v_h]\,,\, w \in \{l,h\}$$

By definition, $\phi_w(\cdot|\rho,\overline{g},r^*) = \bar{F}_w(\cdot|\rho,\overline{g})$, which concludes the proof of Proposition 3.1. ∎

Remarks on Partially Revealing Equilibria.—The critical step in the equilibrium characterization when $\min\left\{n_l^k, n_h^k\right\} \to \infty$ and $\overline{g}\lim \frac{n_h^k}{n_l^k} > 1$ is the finding that there are no atoms in the limit, Lemma 9. The proof of the lemma relies on the fact that when $\overline{g}\lim \frac{n_h^k}{n_l^k} > 1$, then $\mathbb{E}\left[v|x,\, x_{(1)} \leq x;\, \beta^k,\, \boldsymbol{\eta}^k\right]$ is strictly increasing for sufficiently large x. Once we have shown that there are no atoms, we can use essentially standard arguments to characterize the equilibrium outcome; see especially Kremer (2002). If there are no atoms, then the distribution of the winning bid is characterized by the zero profit condition (8), which requires that every price in the support is equal to the expected value conditional on winning at that price in the limit. Equivalently, the expected payoff of the winning bidder converges to zero, that is, the auction becomes "competitive" in the sense of Kremer (2002).

We use this to interpret further the equilibrium outcome. Suppose $\left\{\beta^k, \boldsymbol{\eta}^k\right\}$ is such that $\lim \frac{n_h^k}{n_l^k} = \hat{r}$. Then, the upper bound of the distribution of the winning bid the expected value conditional on the highest signal $\bar{x}$,

$$\frac{v_l + \rho\overline{g}\hat{r}v_h}{1+\rho\overline{g}\hat{r}} = \lim_{k\to\infty} \mathbb{E}[v|\bar{x};\, \boldsymbol{\eta}^k].$$

Let y denote the highest signal among the solicited bidders. The construction of $\phi_w(p|\rho,\overline{g},\hat{r})$ shows that the limit distribution of prices is identical to the limit distribution of $E\left[v|y,\, x_{(1)} \leq y,\, \boldsymbol{\eta}^k\right]$, the posterior of the bidder with the highest signal,

$$\phi_w(p|\rho,\overline{g},\hat{r}) = \lim_{k\to\infty} \Pr_y[\mathbb{E}[v|y,\, x_{(1)} \leq y;\, \boldsymbol{\eta}^k] \leq p|w]. \tag{23}$$

Comparison with a Large Ordinary Common Value Auction.—The ordinary common value auction is nested as a bidding game in which the auctioneer is known to solicit the same number of bidders in each state. Consider a sequence of solicitation strategies $\left\{\boldsymbol{\eta}_0^k\right\}$ with $n_{0l}^k = n_{0h}^k = n_0^k$ for all k, along which the auction becomes large, $\lim n_0^k = \infty$. Let β_0^k be a bidding equilibrium given $\boldsymbol{\eta}_0^k$. Lemma 11 applies to this sequence with ratio $r=1$, so that the distribution of the winning bid is given by $\phi_w(p|\rho,\overline{g},r=1)$ in state w.

Inspection of ϕ_h shows that when $\overline{g}$ increases, $\phi_h(p|\rho,\overline{g},r=1)$ decreases at every p, that is, for $\overline{g}' > \overline{g}$, $\phi_h(\cdot|\rho,\overline{g}',r=1)$ stochastically dominates $\phi_h(\cdot|\rho,\overline{g},r=1)$. Consequently, the expected revenue of the seller increases in the likelihood ratio $\overline{g}$ in state $w=h$. Since the ex ante expected revenue equals the ex ante expected value, this implies that the expected revenue of the seller decreases in the likelihood ratio $\overline{g}$ in state $w=l$.

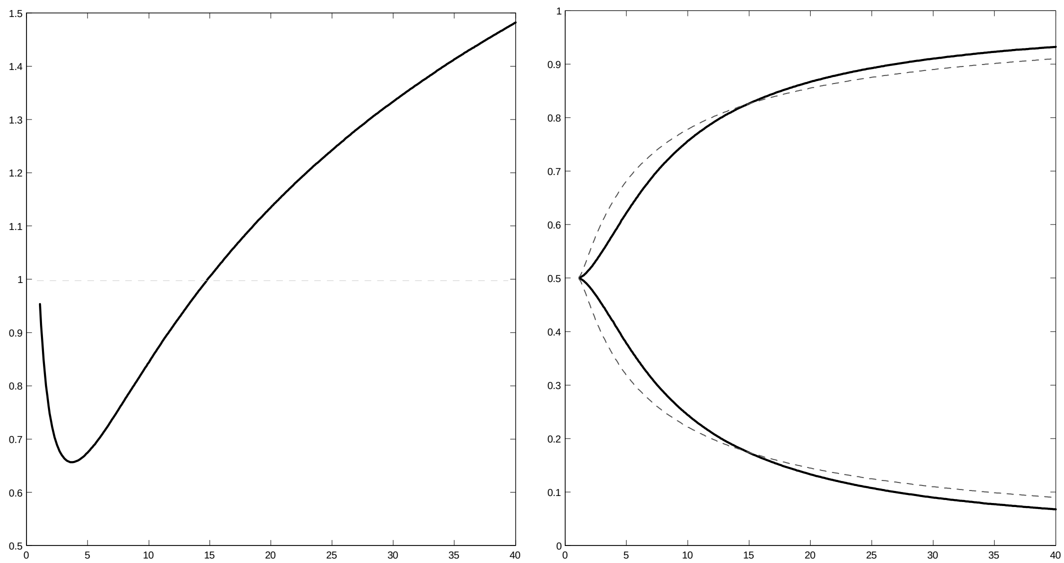

Figure 1: Left Panel: The ratio of the number of sampled bidders, $r^*(\bar{g}, 1)$. Right Panel: Straight black lines are the expected revenues with solicitation, $\bar{E}_S[p|1, \overline{g}, h]$ (top) and $\bar{E}_S[p|1, \overline{g}, l]$ (bottom), Dashed grey lines are the expected revenues of the ordinary common value auction, $\bar{E}_0[p|1, \overline{g}, h]$ (top) and $\bar{E}_0[p|1, \overline{g}, l]$ (bottom).

Note that the limit of the distribution of the winning bid depends only on ρ, $\overline{g}$, and w in both, the ordinary auction and the partially separating outcome of the auction with solicitation. Denote the limits of the expected revenue in each state by $\bar{E}_0[p|\rho, \overline{g}, w]$ and $\bar{E}_S[p|\rho, \overline{g}, w]$, respectively. When $r^*(\rho, \overline{g}) = 1$, then the equilibrium outcomes of the ordinary common value auction is identical to the partially separating equilibrium outcome of the auction with bid solicitation, so that $\bar{E}_0[p|\rho, \overline{g}, w] = \bar{E}_S[p|\rho, \overline{g}, w]$ for $w \in \{l, h\}$. Inspection of $\phi_h(p|\rho, \overline{g}, r)$ shows that it is decreasing in r. Consequently, when $r^*(\rho, \overline{g}) < 1$ (when there is a *solicitation curse*), then there is less information revelation with bidder solicitation, in the sense that the expected revenue in the two states is more similar than in the ordinary auction, $\bar{E}_S[p|\rho, \overline{g}, h] < \bar{E}_0[p|\rho, \overline{g}, h]$ and $\bar{E}_S[p|\rho, \overline{g}, l] > \bar{E}_0[p|\rho, \overline{g}, l]$. When $r^*(\rho, \overline{g}) > 1$ (when there is a *solicitation blessing*), there is more information revelation with bidder solicitation, $\bar{E}_S[p|\rho, \overline{g}, h] > \bar{E}_S[p|\rho, \overline{g}, h]$ and $\bar{E}_S[p|\rho, \overline{g}, l] < \bar{E}_0[p|\rho, \overline{g}, l]$. Figure 1 illustrates the shape of the ratio $r^*(1, \overline{g})$ and compares the expected revenue of each type of seller with and without solicitation. As shown, when $\bar{g}$ is small, $r^*(1, \overline{g}) < 1$ and when $\bar{g}$ is large, $r^*(1, \overline{g}) > 1$.[8]

[8] Numerical analysis shows that $r^*(1, \overline{g}) > 1$ is true for values of $\bar{g}$ up to $\bar{g} = 100,000$. We conjecture that one can find for all ρ a cutoff $\hat{g}(\rho)$ such that $r^*(\rho, \overline{g}) \gtrless 1$ if $\bar{g} \gtrless \hat{g}(\rho)$, but we have not been able to verify this conjecture.

6.2 Proof of Proposition 3.2: Pooling Equilibria

We consider a sequence of equilibria $(\beta^k, \boldsymbol{\eta}^k)$ that satisfy the hypothesis of Proposition 3.2, that is, the distributions F_w and the ratio $\frac{n_h^k}{n_l^k}$ converge, and, in addition, $\frac{g_h(\bar{x})}{g_l(\bar{x})} \lim \frac{n_h^k}{n_l^k} \leq 1$.

We start by showing that if the number of solicited bids is bounded in the limit, then the distribution of the winning bid is degenerate . The lemma includes the trivial equilibrium in which both types of the seller sample a single buyer who bids $\beta^k \equiv 0$.

Lemma 13 *Consider any sequence of strategy profiles* $(\beta^k, \boldsymbol{\eta}^k)$ *such that* $\boldsymbol{\eta}^k$ *is an optimal solicitation strategy given* β^k. *If* $\min\left\{n_l^k, n_h^k\right\}$ *is bounded, then the distribution of the winning bid converges to a degenerate distribution with probability mass one on some number* $C \leq \mathbb{E}_0[v]$ *in both states.*

Proof: Suppose $\lim_{k\to\infty} n_h^k = m < \infty$ (the case where n_l^k is bounded is analogous). Recall that β^k is not necessarily monotone and $\left(\beta^k\right)^{-1}([0,p]) = \left\{x : \beta^k(x) \in [0,p]\right\}$. As $s^k \to 0$, optimality requires that the right side of (4) vanishes, that is,

$$\lim_{k\to\infty} \int_0^{v_h} (G_h((\beta^k)^{-1}([0,p])))^m \left(1 - G_h((\beta^k)^{-1}([0,p]))\right) dp = 0.$$

Thus, the integrand converges to zero almost everywhere. From the monotonicity of G_h, this requires that for some $C \in [0, v_h]$, $\lim_{k\to\infty} G_h((\beta^k)^{-1}(p)) = 1$ if $p > C$ and $= 0$ if $p < C$. Thus, the distribution of the winning bid becomes concentrated at C. Finally, the number C must be below $\mathbb{E}_0[v]$ by individual rationality of buyers and the law of iterated expectations. ∎

We now consider outcome with an unbounded number of solicited bids. Proposition 1 allows us to assume that β^k is nondecreasing. Without loss of generality, we may further simplify the proof by assuming that for all k,

$$\beta^k(\underline{x}) = \lim_{\varepsilon\to 0} \beta^k(\underline{x} + \varepsilon) \text{ and } \beta^k(\bar{x}) = \lim_{\varepsilon\to 0} \beta^k(\bar{x} - \varepsilon).$$

Lemma 14 *Consider any sequence of strategy profiles* $(\beta^k, \boldsymbol{\eta}^k)$ *such that* $\min\left\{n_l^k, n_h^k\right\} \to \infty$. *Suppose that* $\lim \pi_l\left[\beta^k(\bar{x}) | \beta^k, n_l^k\right] > 0$ *and suppose* x^k *is a sequence of signals with* $x^k \leq x_-^k\left(\beta^k(\bar{x})\right)$ *and* $\lim G_l\left(x^k\right)^{n_l^k} > 0$. *If* $\frac{g_h(\bar{x})}{g_l(\bar{x})} \lim \frac{n_h^k}{n_l^k} \leq 1$, *then*

$$\lim_{k\to\infty} E\left[v | \bar{x},\, x_{(1)} \leq x^k;\, \beta^k,\, \boldsymbol{\eta}^k\right] \geq \lim_{k\to\infty} E\left[v | \bar{x},\, \text{win at } \beta^k(\bar{x});\, \beta^k,\, \boldsymbol{\eta}^k\right],$$

with strict inequality if $\overline{g} \lim(n_h^k/n_l^k) < 1$ *and* $\lim G_l\left(x^k\right)^{n_l^k} < 1$.

Proof: We need to show

$$\lim_{k\to\infty} \frac{G_h\left(x^k\right)^{n_h^k-1}}{G_l\left(x^k\right)^{n_l^k-1}} \geq \lim_{k\to\infty} \frac{\pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right]}. \tag{24}$$

(Here, and in the following, we assume that all limits exist, restricting attention to subsequences if necessary.) To simplify the notation, we denote $x_-^k = x_-^k\left(\beta^k\left(\bar{x}\right)\right)$, $z = \frac{g_h(\bar{x})}{g_l(\bar{x})} \lim \frac{n_h^k}{n_l^k}$, and $q = \lim G_l\left(x_-^k\right)^{n_l^k-1}$. The hypothesis $\lim \pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right] > 0$ implies that $q > 0$. If $q = 1$, then

$$\lim_{k\to\infty} \frac{\pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right]} = 1.$$

If $q < 1$, then Lemma 3 and $G_w\left(x_+^k\left(\beta^k\left(\bar{x}\right)\right)\right) = 1$ for all k imply

$$\lim_{k\to\infty} \pi_w\left[\beta^k\left(\bar{x}\right)|\beta^k, n_w^k\right] = \lim_{k\to\infty} \frac{1 - G_w\left(x_-^k\right)^{n_w^k}}{n_w^k\left(1 - G_w\left(x_-^k\right)\right)}.$$

By Lemma 5,

$$\lim_{k\to\infty} G_h\left(x_-^k\right)^{n_h^k} = q^z.$$

From $\lim G_l\left(x_-^k\right)^{n_l^k} = q \in (0,1)$, $\lim n_l^k\left(1 - G_l\left(x_-^k\right)\right) = -\ln q$. Combining the previous observations,

$$\lim_{k\to\infty} \frac{\pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right]}{\pi_l\left[\beta^k\left(\bar{x}\right)|\beta^k, n_l^k\right]} = \frac{\left(1-q^z\right)\ln q}{\left(1-q\right)\ln q^z} = \frac{1-q^z}{\left(1-q\right)z}. \tag{25}$$

Let $\hat{q} = \lim G_l\left(x^k\right)^{n_l^k-1}$ for the sequence x^k. By hypothesis, $\hat{q} > 0$. Since $x^k \leq x_-^k$, $\hat{q} \leq q \leq 1$. By Lemma 5, $\lim G_h\left(x_-^k\right)^{n_h^k-1} = \hat{q}^z$. Since $\hat{q} \leq q$ and $z \leq 1$,

$$\lim_{k\to\infty} \frac{G_h\left(x^k\right)^{n_h^k-1}}{G_l\left(x^k\right)^{n_l^k-1}} = \hat{q}^{z-1} \geq q^{z-1}.$$

Therefore, (25) implies that the Inequality (24) holds if

$$q^{z-1} \geq \frac{1-q^z}{\left(1-q\right)z}.$$

This inequality holds, because $q^{z-1}\left(1-q\right)z - \left(1-q^z\right)$ is decreasing in q, and is equal to zero when $q = 1$.

Finally, if $\hat{q} < 1$ and $z < 1$, then $\hat{q}^{z-1} > q^{z-1}$, which implies that the Inequality (24) is strict. ∎

Proof of Proposition 3.2: Lemma 13 implies the Proposition if the number of solicited bidders stays bounded. So, suppose $\min\left\{n_l^k, n_h^k\right\}$ is unbounded.

Given any $\varepsilon < 1/2$, define x_ε^k and $x_{1-\varepsilon}^k$ as the corresponding quantiles,

$$G_l\left(x_\varepsilon^k\right)^{n_l^k} = \varepsilon \text{ and } G_l\left(x_{1-\varepsilon}^k\right)^{n_l^k} = 1-\varepsilon.$$

We show that $\overline{g}\lim\left(n_h^k/n_l^k\right) = z \leq 1$ implies that

$$\lim_{k\to\infty}\beta^k\left(x_\varepsilon^k\right) = \lim_{k\to\infty}\beta^k\left(x_{1-\varepsilon}^k\right) \tag{26}$$

for all ε. By contradiction: Suppose for some ε, $\lim\beta^k\left(x_\varepsilon^k\right) < \lim\beta^k\left(x_{1-\varepsilon}^k\right)$. We can choose b_1 and b_2 such that $\lim\beta^k\left(x_\varepsilon^k\right) < b_1 < b_2 < \lim\beta^k\left(x_{1-\varepsilon}^k\right)$. Because the grid becomes fine, there is some sequence b^k with $b^k \in (b_1, b_2)$ such that there is no atom at b^k in the limit,[9]

$$\lim_{k\to\infty}\left(G_l\left(x_+^k\left(b^k\right)\right)^{n_l^k} - G_l\left(x_-^k\left(b^k\right)\right)^{n_l^k}\right) = 0. \tag{27}$$

The hypothesis $\beta^k\left(x_\varepsilon^k\right) < \beta^k\left(x_{1-\varepsilon}^k\right)$ and the choice of x_ε^k imply that $\lim\pi_h\left[\beta^k\left(\bar{x}\right)|\beta^k, n_h^k\right] > 0$. Hence, the zero profit condition (8) requires that

$$\lim_{k\to\infty} E\left[v|\bar{x},\text{ win at }\beta^k\left(\bar{x}\right);\beta^k,\boldsymbol{\eta}^k\right] = \lim_{k\to\infty}\beta^k\left(\bar{x}\right). \tag{28}$$

Similarly, $\lim\pi_h\left[b^k|\beta^k, n_h^k\right] > 0$. Hence, Equation (6) requires

$$\lim_{k\to\infty}\mathbb{E}\left[v|\bar{x},\text{ win at } b^k;\beta^k,\boldsymbol{\eta}^k\right] \leq \lim_{k\to\infty} b^k. \tag{29}$$

By construction, $x_+^k\left(b^k\right) \leq x_+^k\left(\beta^k\left(\bar{x}\right)\right)$ and $\lim G_l\left(x_+^k\left(b^k\right)\right)^{n_l^k} \geq \varepsilon$. Hence, (27) and Lemma 14 imply that

$$\begin{aligned}\lim_{k\to\infty}\mathbb{E}\left[v|\bar{x},\text{ win at } b^k;\beta^k,\boldsymbol{\eta}^k\right] &= \lim_{k\to\infty}\mathbb{E}\left[v|\bar{x},\, x_{(1)} \leq x_+^k\left(b^k\right);\beta^k,\boldsymbol{\eta}^k\right]\\ &\geq \lim_{k\to\infty}\mathbb{E}\left[v|\bar{x},\text{ win at }\beta^k\left(\bar{x}\right);\beta^k,\boldsymbol{\eta}^k\right].\end{aligned}$$

Thus, (28) and (29) require $\lim b^k \geq \lim\beta^k\left(\bar{x}\right)$, a contradiction.

[9] See, for example, the proof of Lemma 9, Step 3.

Thus, (26) holds for all ε. Therefore, the distribution of the winning bid converges to $C = \lim \beta^k \left(x^k_\varepsilon\right)$, for any $\varepsilon > 0$. Individual rationality of buyers' bids and the law of iterated expectations together implies that $C \leq \rho_l v_l + \rho_h v_h$. ∎

Atoms.—We can strengthen the result if the set of feasible bids is $[0, v_h]$. As before, given some $0 < \varepsilon < \frac{1}{2}$, define x^k_ε and $x^k_{1-\varepsilon}$ as the corresponding quantiles,

$$G_l \left(x^k_\varepsilon\right)^{n^k_l} = \varepsilon \text{ and } G_l \left(x^k_{1-\varepsilon}\right)^{n^k_l} = 1 - \varepsilon.$$

Lemma 15 Equilibrium Atom *Suppose $\Delta^k = 0$ for all k. Consider any sequence of strategy profiles $(\beta^k, \boldsymbol{\eta}^k)$ such that $\min\left\{n^k_l, n^k_h\right\} \to \infty$ and $\overline{g} \lim \left(n^k_h / n^k_l\right) < 1$. Then, for all ε there is some $K(\varepsilon)$ such that $k \geq K(\varepsilon)$ implies*

$$\beta^k \left(x^k_\varepsilon\right) = \beta^k \left(x^k_{1-\varepsilon}\right).$$

The lemma implies that when $\Delta^k = 0$, then there exists no bidding equilibrium in which β^k is strictly increasing when $\overline{g}\left(n^k_h / n^k_l\right) < 1$ and the number of bidders is sufficiently large. Furthermore, there is a large atom at the top in the limit, in the sense that the winning bid is equal to $\beta^k \left(x^k_\varepsilon\right)$ with probability converging to one in both states.

Proof: By contradiction. Suppose that for some ε and some sequence of equilibria

$$\beta^k \left(x^k_\varepsilon\right) < \beta^k \left(x^k_{1-\varepsilon}\right),$$

for all k. Since $\Delta^k = 0$, we can pick some b^k for each k such that $\beta^k \left(x^k_\varepsilon\right) < b^k < \beta^k \left(x^k_{1-\varepsilon}\right)$ and such that there is not atom at b^k, $G_l \left(x^k_+ \left(b^k\right)\right)^{n^k_l} = G_l \left(x^k_- \left(b^k\right)\right)^{n^k_l}$. Since $G_l \left(x^k_+ \left(b^k\right)\right)^{n^k_l} \in (\varepsilon, 1-\varepsilon)$ and $\overline{g} \lim \left(n^k_h / n^k_l\right) < 1$, Lemma 14 and the zero profit condition imply

$$\begin{aligned}
\lim_{k\to\infty} b^k &\geq \lim_{k\to\infty} \mathbb{E}\left[v | \bar{x},\, x_{(1)} \leq x^k_+ \left(b^k\right);\, \beta^k,\, \boldsymbol{\eta}^k\right] \\
&> \lim_{k\to\infty} \mathbb{E}\left[v | \bar{x}, \text{ win at } \beta^k(\bar{x});\, \beta^k,\, \boldsymbol{\eta}^k\right] = \lim_{k\to\infty} \beta^k (\bar{x}).
\end{aligned}$$

This is in contradiction to $b^k < \beta^k \left(x^k_{1-\varepsilon}\right) \leq \beta^k (\bar{x})$. ∎

Remarks on Pooling Equilibria.—The critical step in the equilibrium characterization when $\min\left\{n^k_l, n^k_h\right\} \to \infty$ and $\overline{g} \lim \frac{n^k_h}{n^k_l} \leq 1$ is the finding that the expected value conditional on winning at $\beta^k(\bar{x})$ is smaller than the expected value conditional on winning with a bid just below $\beta^k(\bar{x})$, if there is no atom at that bid, Lemma 14. This implies that the distribution of the winning bid must become degenerate because otherwise there would be bids below $\beta^k(\bar{x})$ at which the winning probability is positive and the expected value conditional on winning is strictly larger than $\beta^k(\bar{x})$, as shown in the proof of the Proposition.

The distribution of the winning bid may converge to a price strictly below the ex ante expected value. Thus, the expected value conditional on winning may be strictly higher than the bid, even in the limit, meaning that the bidders as a group may expect strictly positive profits and the auction does not become "competitive." We now turn to the question whether such pooling equilibria in fact exist.

7 Existence of Equilibrium with Grid

We now consider the auction when the relevant set of feasible bids is a grid. We show that a sequence of partially revealing equilibria always exists. In addition, a sequence of nontrivial pooling equilibria exists under some additional conditions on the distribution of signals. We do not know whether pooling equilibria exist for all specifications of the model. Finally, we comment on the importance of the grid for the existence results.

7.1 Existence of Pooling Equilibria

We construct nontrivial pooling equilibria in which the number of bidders is large when solicitation costs are small. Pooling equilibria in which the seller solicits only a single bid always exist.

Discrete Signals.—We assume that signals are essentially finite. Specifically, the range of the signal values $[\underline{x}, \bar{x}]$ is divided into m subintervals

$$[\underline{x}, \epsilon], (\epsilon, 2\epsilon], \cdots, (\bar{x} - \epsilon, \bar{x}].$$

The density functions g_w, $w = h, l$, are step functions that are constant over each of these intervals and jump upwards at the boundaries. Consequently, the likelihood ratio $\frac{g_h(x)}{g_l(x)}$ is a step function as well, so there are at most m different likelihood ratios.[10] Thus, as far as the information is concerned, this is a discrete signal structure with m values; the continuum is kept only for purification purposes. As before, the likelihood ratio $\frac{g_h(x)}{g_l(x)}$ is nondecreasing, so that low signals are less favorable than high signals.

Following are two conditions on the signal structure that we refer to in the

[10]The important assumption is the finiteness of the set of values that the likelihood ratio takes on. Density functions that are also step functions are consistent with that assumption but are not necessary.

subsequent analysis. First,

$$\frac{1}{G_l(\bar{x}-\epsilon)}\frac{1-G_l(\bar{x}-\epsilon)}{1-G_h(\bar{x}-\epsilon)}<1, \tag{30}$$

which is equivalent to $\frac{1}{G_l(\bar{x}-\epsilon)}<\frac{g_h(\bar{x})}{g_l(\bar{x})}$. Second,

$$\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\frac{G_l(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}\leq 1. \tag{31}$$

The first condition is naturally satisfied if ϵ is not too large, since then $G_l(\bar{x}-\epsilon)$ is near 1, while $\frac{g_h(\bar{x})}{g_l(\bar{x})}>1$. The second condition can be thought of as a strengthening of the increasing likelihood ratio requirement at the top of the support. The condition is in particular implied by the increasing failure ratio property (Herrera and Hörner; 2012),[11]

$$\frac{g_h(\underline{x})}{g_l(\underline{x})}\leq\frac{\frac{g_h(\underline{x}+\epsilon)}{G_h(\underline{x}+\epsilon)}}{\frac{g_l(\underline{x}+\epsilon)}{G_l(\underline{x}+\epsilon)}}\leq\cdots\leq\frac{\frac{g_h(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)}}{\frac{g_l(\bar{x}-\epsilon)}{G_l(\bar{x}-\epsilon)}}\leq\frac{\frac{g_h(\bar{x})}{G_h(\bar{x})}}{\frac{g_l(\bar{x})}{G_l(\bar{x})}}.$$

An example of a (continuous) distribution of signals with an increasing failure ratio property is $G_w(x)=x^{z_w}$, with $z_h>z_l$. For this distribution, the failure ratio is constant.

To construct the pooling equilibrium, we study the following auxiliary game.
Auxiliary Game A: Let $\Gamma^A\left(s,P_\Delta|\underline{b},\bar{b}\right)$ denote a variation of the game in which buyers with signal $x\leq\bar{x}-\epsilon$ cannot bid more than $\underline{b}$ and buyers with signal $x>\bar{x}-\epsilon$ must bid $\bar{b}$, with $\underline{b}<\bar{b}\in P_\Delta$. Therefore, bidding strategies must satisfy

$$\beta(x)\begin{cases}=\bar{b} & \text{if} \quad x>\bar{x}-\epsilon,\\ \leq\underline{b} & \text{if} \quad x\leq\bar{x}-\epsilon.\end{cases} \tag{32}$$

The seller's solicitation strategy is unconstraint and the game is the same as the original game in all other respects. A strategy profile $(\beta,\boldsymbol{\eta})$ is an equilibrium $\Gamma^A\left(s,P_\Delta|\underline{b},\bar{b}\right)$ if (i) $\boldsymbol{\eta}$ is an optimal solicitation strategy for the seller given β, and (iia) given $\boldsymbol{\eta}$, the strategy $\beta(x)$ is a best response for any $x\leq\bar{x}-\epsilon$, subject to the constraint $\beta(x)\leq\underline{b}$, (iib), $\beta(x)=\bar{b}$ if $x>\bar{x}-\epsilon$.

When $\Delta>0$, the auxiliary game is a finite Bayesian game. Existence of an equilibrium is proved in the Appendix and relies on Proposition 1. The proof follows Athey (2001).

[11] From the displayed equation, $\frac{g_h(\bar{x}-\epsilon)}{g_l(\bar{x}-\epsilon)}\frac{G_l(\bar{x}-\epsilon)}{G_h(\bar{x}-\epsilon)}\leq\frac{g_h(\bar{x})}{g_l(\bar{x})}$. The second condition now follows because $\frac{g_h(\bar{x})}{g_l(\bar{x})}=\frac{1-G_h(\bar{x}-\varepsilon)}{1-G_l(\bar{x}-\varepsilon)}<\frac{\ln G_h(\bar{x}-\varepsilon)}{\ln G_l(\bar{x}-\varepsilon)}$, since $G_h(\bar{x}-\varepsilon)<G_l(\bar{x}-\varepsilon)$ and $\frac{1-z}{\ln z}$ is decreasing in z.

Lemma 16 *The Auxiliary Game A has an equilibrium if the grid size* $\Delta > 0$.

Suppose that all bidders use a bidding strategy β^k as in (32). The following lemma calculates limiting expected values conditional on winning and is proved in the Appendix.

Lemma 17 *There are numbers* v_1^*, v_2^*,$v_3^* < \rho_l v_l + \rho_h v_h$ *such that the following is true. If* $s^k \to 0$ *and* $\boldsymbol{\eta}^k = (\eta_l^k, \eta_h^k)$ *is an optimal solicitation strategy for some sequence of bidding strategies* β^k *that satisfy (32) for any* $\underline{b}$, $\bar{b}$, *then* $\min\left\{n_l^k, n_h^k\right\} \to \infty$ *and*

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\ \textit{win at } \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] &= \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon,\ \textit{win at } \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] &\leq v_1^* < \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x},\ \textit{win at } b > \bar{b};\ \beta^k,\ \boldsymbol{\eta}^k\right] &\leq v_2^* < \rho_l v_l + \rho_h v_h, \\
\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon,\ \textit{win at } b \in \left(\underline{b}, \bar{b}\right);\ \beta^k, \boldsymbol{\eta}^k\right] &\leq v_3^* < \rho_l v_l + \rho_h v_h.
\end{aligned}$$

We discuss how the lemma follows from the seller's optimality conditions. First, it is immediate from the construction of β^k that when the solicitation costs are small, both types of the seller optimally solicit an unbounded number of bidders.

For the first equation in the lemma, note that the increasingly large number of bidders implies that in the limit the auction ends with probability converging to one with some buyer with a signal $x > \bar{x} - \epsilon$ winning the auction with a bid $\bar{b}$. Hence, this event ceases to contain information. Therefore, the expected value conditional on that event converges to the prior expected value. The second equation is an immediate consequence of the fact that the signal $x > \bar{x} - \epsilon$ is more favorable than $x \leq \bar{x} - \epsilon$.

We illustrate the third equation of the lemma with an example. Suppose that the bidding strategy is simply

$$\beta^k(x) = \begin{cases} \bar{b} & \text{if} \quad x > \bar{x} - \epsilon, \\ \underline{b} & \text{if} \quad x \leq \bar{x} - \epsilon, \end{cases}$$

the optimal solicitation strategy is pure, and the marginal benefit of an additional bidder is exactly equal to its cost ,

$$\begin{aligned}
\left(G_l(\bar{x}-\epsilon)\right)^{n_l^k}\left(1 - G_l(\bar{x}-\epsilon)\right)\left(\bar{b} - \underline{b}\right) &= s^k, \\
\left(G_h(\bar{x}-\epsilon)\right)^{n_h^k}\left(1 - G_h(\bar{x}-\epsilon)\right)\left(\bar{b} - \underline{b}\right) &= s^k;
\end{aligned}$$

that is, we can ignore the integer problem. Setting the two optimality conditions equal, cancelling $\left(\bar{b}-\underline{b}\right)$, and taking the natural logarithm shows

$$n_l^k \ln G_l\left(\bar{x}-\epsilon\right)+\ln\left(1-G_l\left(\bar{x}-\epsilon\right)\right)=n_h^k \ln G_h\left(\bar{x}-\epsilon\right)+\ln\left(1-G_h\left(\bar{x}-\epsilon\right)\right).$$

When $n_l^k, n_h^k \rightarrow \infty$, the first term on each side dominates the second term. Thus, the ratio of solicited bidders converges,

$$\lim_{k\rightarrow\infty}\frac{n_h^k}{n_l^k}=\frac{\ln G_l\left(\bar{x}-\epsilon\right)}{\ln G_h\left(\bar{x}-\epsilon\right)}.$$

This ratio is smaller than one, so that being solicited is bad news. Moreover, $\frac{g_h(\bar{x})}{g_l(\bar{x})}\frac{\ln G_l(\bar{x}-\epsilon)}{\ln G_h(\bar{x}-\epsilon)}<1$, which follows from $\frac{g_h(\bar{x})}{g_l(\bar{x})}=\frac{1-G_h(\bar{x}-\epsilon)}{1-G_l(\bar{x}-\epsilon)}$ and $\frac{1-z}{\ln z}$ being decreasing in z. Hence,

$$\frac{g_h\left(\bar{x}\right)}{g_l\left(\bar{x}\right)}\lim_{k\rightarrow\infty}\frac{n_h^k}{n_l^k}<1. \tag{33}$$

This implies that conditional on signal $\bar{x}$ and conditional on being solicited, the expected value is smaller than the prior expected value. Since a buyer bidding above $\bar{b}$ wins in either state, winning contains no information. Thus, (33) implies the third equation. The Appendix shows that (33) holds also when the bidding strategy may involve bids below $\underline{b}$ and when the integer problem is not ignored.

The fourth equation of the lemma is proved similarly. This part of the proof uses the Assumptions (30) and (31).

The following proposition brings the main result of this section. To state the Proposition, pick any $\underline{b}$ and $\bar{b}$ that satisfy

$$\max\left\{v_1^*, v_2^*, v_3^*\right\}<\underline{b}<\bar{b}<\rho_l v_l+\rho_h v_h. \tag{34}$$

Proposition 4 Existence of Pooling Equilibrium *Suppose that the signals satisfy conditions (30) and (31), $\Delta^k>0$, and $\underline{b}$, $\bar{b}\in P_{\Delta^k}$ satisfy (34). For any vanishing sequence of solicitation costs $\left\{s^k\right\}$, with $s^k\rightarrow 0$, there exists a sequence of equilibria $\left(\beta^k, \boldsymbol{\eta}^k\right)$ such that*

- $\min\left\{n_l^k, n_h^k\right\}\rightarrow\infty$.
- $\beta^k\left(x\right)=\bar{b}$ *for all* $x>\bar{x}-\epsilon$ *and* k *sufficiently large.*

Thus, for this sequence of equilibria, the winning bid converges to $\bar{b}$ almost surely.

Proof of Proposition 4:

By Lemma 16, the Auxiliary Game $\Gamma^A\left(s^k, P_{\Delta^k}|\underline{b},\bar{b}\right)$ has an equilibrium $\left(\beta^k, \boldsymbol{\eta}^k\right)$ whenever $\Delta^k>0$ by Lemma 16.

We show that $(\beta^k, \boldsymbol{\eta}^k)$ is an equilibrium of the original game for s^k sufficiently small by proving that the two additional constraints on the bidding strategies do not bind. To do so, we utilize the characterization from Lemma 17.

Step 0. $\min\left\{n_l^k, n_h^k\right\} \to \infty$.

From Lemma 17.

Step 1. Bidding $\beta^k = \bar{b}$ is optimal if $x > \bar{x} - \epsilon$.

(i) Bidding $b > \bar{b}$ is unprofitable. By the choice of $\bar{b} > v_2^*$ and Lemma (17),

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k,\, \boldsymbol{\eta}^k\right] < \bar{b}. \tag{35}$$

Thus, there is some K_1 such that bidding $b > \bar{b}$ is strictly unprofitable for all $k \geq K_1$.

(ii) Bidding $b < \bar{b}$ is unprofitable. First, by Lemma 17 and the choice of $\bar{b}$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}, \text{ win at } \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] = \rho_l v_l + \rho_h v_h > \bar{b}. \tag{36}$$

For any $b < \bar{b}$, Lemma 3 implies

$$\lim_{k\to\infty} \frac{\pi_w\left[\bar{b}|\beta^k, \eta_w^k\right]}{\pi_w\left[b|\beta^k, \eta_w^k\right]} \geq \lim_{k\to\infty} \frac{\frac{1}{n_w^k}\frac{1}{1-G_w(\bar{x}-\epsilon)}\left(1-\left(G_w\left(\bar{x}-\epsilon\right)\right)^{n_w^k}\right)}{\left(G_w\left(\bar{x}-\epsilon\right)\right)^{n_w^k-1}} = \infty. \tag{37}$$

where the last equality follows from $n_w^k \to \infty$. By (36), the payoff conditional on winning at $\bar{b}$ is bounded away from 0. It now follows from (37) that there is some K_2 such that for all $k \geq K_2$, the payoff from bidding $b < \bar{b}$ is an arbitrarily small fraction of the payoff of bidding $\bar{b}$, so that undercutting $\bar{b}$ is unprofitable for $x > \bar{x} - \epsilon$. □

Step 2. Bidding $b > \underline{b}$ is unprofitable for $x \leq \bar{x} - \epsilon$.

(i) Bidding $b > \bar{b}$ is unprofitable. Obviously, for all k,

$$\mathbb{E}\left[v|\bar{x}-\epsilon, \text{ win at } b > \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] < \mathbb{E}\left[v|\bar{x}, \text{ win at } b > \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right].$$

From Lemma (17) and $\bar{b} < v_2^*$, the limit of the right side is strictly below $\bar{b}$. Therefore,

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon, \text{ win at } b > \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] < \bar{b}.$$

Of course, for $x \leq \bar{x} - \epsilon$,

$$\mathbb{E}\left[v|x, \text{ win at } b > \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] \leq \mathbb{E}\left[v|\bar{x}-\epsilon, \text{ win at } b > \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right].$$

Hence, there is some K_3 such that for all $k \geq K_3$ and $x \leq \bar{x} - \epsilon$

$$\mathbb{E}\left[v|x, \text{ win at } b_i > \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] < \bar{b}.$$

(ii) Bidding $\bar{b}$ is unprofitable. By Lemma 17 and the choice of $\bar{b} < v_1^*$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon, \text{ win at } \bar{b};\, \beta^k, \boldsymbol{\eta}^k\right] < \bar{b}.$$

Thus, there is some K_4, such that bidding $\bar{b}$ is unprofitable for $x \leq \bar{x}-\epsilon$ when $k \geq K_4$.

(iii) Bidding $b \in \left(\underline{b}, \bar{b}\right)$ is unprofitable. From Lemma (17) and the choice of $\bar{b}$,

$$\lim_{k\to\infty} \mathbb{E}\left[v|\bar{x}-\epsilon, \text{ win at } b_i \in \left(\underline{b}, \bar{b}\right);\, \beta^k, \boldsymbol{\eta}^k\right] \leq v_3^* < \underline{b}.$$

Thus, there is some K_5 such that for all $k \geq K_5$ bidding $b \in \left(\underline{b}, \bar{b}\right)$ is unprofitable for all $x \leq \bar{x}-\epsilon$. □

Let $K = \max\{K_1, .., K_5\}$. Step 1 and Step 2 imply that the additional constraints of the auxiliary game do not bind when $k \geq K$. Thus, $(\beta^k, \boldsymbol{\eta}^k)$ is an equilibrium of the original game for $k \geq K$. For $k < K$, we can pick any equilibrium, including the trivial equilibrium. By construction, $\beta^k(x) = \bar{b}$ for all $x > \bar{x}-\epsilon$ and $k \geq K$. From Step 0, $\min\left\{n_l^k, n_h^k\right\} \to \infty$. This proves the Proposition. ∎

7.2 Existence of Partially Revealing Equilibria

We construct a separating sequence of equilibria. For the construction, we utilize the following auxiliary game.

Auxiliary Game B: Let $\Gamma^B\left(s, P_\Delta|\underline{n}_l, \underline{r}\right)$ denote a variation of the game in which the two seller types are represented by separate players who are moving in sequence. The game is the same as the original game in all respects except (i) type l chooses a number n_l first and type h chooses a number n_h afterwards (without observing n_l), and (ii) we restrict the number of actually solicited bidders to be $\hat{n}_l = \max\{n_l, \underline{n}_l\}$ and $\hat{n}_h = \max\{\underline{r}\hat{n}_l, n_h\}$. An equilibrium (β, η_l, η_h) of the Auxiliary Game is defined as usual: β is a bidding equilibrium given the distribution of solicited bidders implied by the solicitation strategy (η_l, η_h) and the solicitation strategy (η_l, η_h) maximizes each seller's profit. Note that the sets of equilibria of $\Gamma^B\left(s, P_\Delta|1, 0\right)$ and $\Gamma\left(s, P_\Delta\right)$ are identical since the constraints do not bind.

When $\Delta > 0$, the auxiliary game is again a finite Bayesian game and an equilibrium exists. The proof is analogous to the proof of Lemma 16 and omitted.

Lemma 18 *The Auxiliary Game B has an equilibrium whenever the grid size $\Delta > 0$.*

We characterize equilibrium outcomes of the Auxiliary Game B.

Lemma 19 *Given* $(s^k, \Delta^k) \to (0,0)$*, with* $s^k > 0$ *and* $\Delta^k \geq 0$*, let* $\underline{n}_l^k$ *and* $\underline{r}$ *be such that* $\underline{n}_l^k = \frac{1}{\sqrt{s^k}}$ *and* $\frac{1}{\overline{g}} < \underline{r} < r^*(\rho, \overline{g})$*. If* $(\beta^k, \boldsymbol{\eta}^k)$ *is an equilibrium of the Auxiliary Game* $\Gamma^B\left(s, P_{\Delta^k} | \underline{n}_l^k, \underline{r}\right)$*, then* $n_l^k > \underline{n}_l^k$ *for large* k*,* $\lim \frac{n_h^k}{n_l^k} = r^*(\rho, \overline{g})$ *and* $\lim F_w\left(p|\beta^k, \eta_w^k\right) = \bar{F}_w\left(p|\rho, \overline{g}\right)$*.*

Proof: Let $r = \lim_{k\to\infty} \frac{\hat{n}_h^k}{\hat{n}_l^k}$. The choices of $\underline{n}_l^k$ and $\underline{r}$ imply $\min\left\{\hat{n}_l^k, \hat{n}_h^k\right\} \to \infty$ and $\overline{g} \lim\left(\hat{n}_h^k / \hat{n}_l^k\right) > 1$.

Hence, Lemma 11 implies that $\lim F_w\left(p|\beta^k, \hat{n}_w^k\right) = \phi_w\left(p|\rho, \overline{g}, r\right)$, for all p and $w = l, h$.

Step 1. For k sufficiently large, $n_l^k > \underline{n}_l^k$.

Proof of Step 1: By choice of $\underline{r}$ and by the argument from Lemma 12, $\frac{1}{\overline{g}} < \lim_{k\to\infty} \frac{\hat{n}_h^k}{\hat{n}_l^k} < \infty$. Hence, ϕ_l is not degenerate. Let m_l^k denote the unconstrained optimal solicitation strategy given β^k. By Lemma 7, m_l^k satisfies

$$\lim_{k\to\infty} m_l^k s^k = -\int_0^{v_h} \phi_l\left(p|\rho, \overline{g}, r\right) \ln\left(\phi_l\left(p|\rho, \overline{g}, r\right)\right) dp > 0.$$

Since $\underline{n}_l^k s^k = \sqrt{s^k} \to 0$, $\lim m_l^k s^k > 0$ implies $\lim \frac{\underline{n}_l^k}{m_l^k} = 0$, so that $m_l^k > \underline{n}_l^k$ sufficiently deep into the sequence. Thus, $n_l^k > \underline{n}_l^k$, as claimed. □

Step 2. $\lim \hat{n}_h^k / \hat{n}_l^k = r^*(\rho, \overline{g})$.

Proof of Step 2: Denote by m_h^k the unconstrained optimal solicitation strategy given β^k. If $m_h^k > \underline{r}\hat{n}_l^k$, then the constraint does not bind, that is, $n_h^k > \underline{r}\hat{n}_l^k$. Thus, by Step 1, $\hat{n}_h^k$ and $\hat{n}_l^k$ are both optimal given β^k, so that Lemma 12 requires that $\lim \frac{\hat{n}_h^k}{\hat{n}_l^k} = r^*(\rho, \overline{g})$. □

We now show that $m_h^k > \underline{r}\hat{n}_l^k$. If $m_h^k \leq \underline{r}\hat{n}_l^k$, then the strict concavity of the seller's optimization implies that $\hat{n}_h^k = \underline{r}\hat{n}_l^k$. By $\frac{1}{\overline{g}} < \underline{r} < r^*(\rho, \overline{g})$ and Lemma 8, it follows that $J(\underline{r}; \rho, \overline{g}) < 0$. Equation (52) from the proof of Lemma 12 shows that $J(\underline{r}; \rho, \overline{g}) < 0$ implies

$$\lim_{k\to\infty} \hat{n}_h^k \left(\mathbb{E}_h\left[p|\beta^k, \hat{n}_h^k + 1\right] - \mathbb{E}_h\left[p|\beta^k, \hat{n}_h^k\right]\right) > \lim_{k\to\infty} \hat{n}_h^k s^k.$$

The incremental benefit of an additional bidder is strictly larger than the solicitation cost when only $\hat{n}_h^k$ bidders are solicited. Hence, the unconstrained optimal number $m_h^k > \underline{r}\hat{n}_l^k$, as claimed. ∎

Lemma 19 shows that for suitably chosen parameters, all sequences of equilibria of the auxiliary game remain equilibria in the original game. Lemma 18 implies there exists a sequence of equilibria in the auxiliary game whenever the set of feasible bids is finite. Together, the two lemmas imply the existence of a sequence of partially revealing equilibria when the set of feasible bids is a sufficiently fine grid.

Proposition 5 *Consider a sequence of games* $\Gamma\left(s^k, P_\Delta\right)$*, where* $\left(s^k, \Delta^k\right) \to 0$*, with* $s^k > 0$ *and* $\Delta^k > 0$*. There exists a sequence of equilibria* $\left(\beta^k, \boldsymbol{\eta}^k\right)$ *such that* $\lim_{k\to\infty} \frac{n_h^k}{n_l^k} = r^*\left(\rho, \overline{g}\right)$ *and* $\lim_{k\to\infty} F_w\left(p|\beta^k, \eta_w^k\right) = \bar{F}_w\left(p|\rho, \overline{g}\right)$*.*

7.3 Existence without Grid

We use the finiteness of the relevant set of feasible bids to prove existence of equilibrium in the auxiliary games. This is the only place where we use the grid. The characterization results in Lemmas 17 and 19 hold also without the grid. Therefore, if we could prove existence of equilibrium for the auxiliary games without the grid, then we could also drop the requirement that $\Delta^k > 0$ from Propositions 4 and 5.

The difficulty for showing existence without a grid is the presence of atoms in equilibrium, which imply that buyers' *equilibrium* payoffs can be discontinuous in their bids. In particular, we cannot argue that the limit of a sequence of equilibria for a vanishingly small grid is an equilibrium of the continuum case. The reason is that there may be atoms in the limit that are absent in the sequence. To illustrate the problem, consider a sequence of games with grid P_{Δ^k} and suppose that along the sequence bidders bid either a constant b or $b+\Delta^k$, depending on whether their signal is below or a above some threshold $\hat{x}$. The pointwise limit strategy as $\Delta^k \to 0$ would be that all bidders bid the constant b. However, this bidding strategy would imply a strictly lower winning probability for buyers who bid $b+\Delta^k$ along the sequence and a strictly higher winning probability for buyers bidding b. Thus, the limit strategy may not be an equilibrium of the game with a continuum of bids, even though the elements of the sequence may have been.[12]

A solution to the existence problem without a grid is to change the tie-breaking rule, as suggested by Jackson, Simon, Swinkels, and Zame (2002). Specifically, consider the following extension: Buyers submit two numbers, the first interpreted as a bid (just as before) and the second number interpreted as eagerness to trade. If there is a unique highest bid, the seller chooses to buy from that bidder. When several bids are tied, the seller may choose among the buyers based on the expressed eagerness. Extending our model in this way solves the existence problem, because the limit of a sequence of equilibrium strategies for a vanishingly small grid corresponds to an equilibrium of the extended game with a continuum of bids. For instance,

[12] There is no such problem for the seller's strategy because of the continuity of the seller's payoffs in β and η. If $\left(\beta^k, \eta^k\right)$ converge pointwise to $\left(\beta^*, \eta^*\right)$, and if η^k is an optimal solicitation strategy given β^k, then η^* is an optimal solicitation strategy given β^*.

in the example from the last paragraph, one may specify as the limit strategy of the extended game that buyers bid b for all signals. Buyers with signals above the threshold (who bid $b+\Delta^k$ along the sequence) all express the same eagerness, say e_h, and buyers with signals below the threshold (who bid b along the sequence) express a different eagerness, say e_l. If multiple bidders are tied at b, then the seller picks first among those bidders who express e_h, choosing randomly if there are multiple such bidders; if no bidder expressed e_h, the seller chooses randomly among bidders expressing e_l (and, finally, choosing bidders who expressed anything else last). This limit strategy preserves the winning probabilities, and, hence, the payoffs in a continuous way. Thus, if the elements of the described sequence of bidding strategies each constitute an equilibrium, so would the limit.

The sketched extension may be especially appealing when our model is interpreted as an informal trading environment, such as the introductory example of a borrower asking for quotes from a number of lenders. Here, the seller may freely choose which of similar quotes to accept and the way in which the seller chooses to break ties may depend on parts of the price quote that are not modeled or it may depend on extraneous information from the communication between the seller and the buyer.

8 Discussion and Conclusion

8.1 Information Aggregation

For a common values auction environment, Wilson (1977) and Milgrom (1979) derived conditions on the informativeness of the signals under which the price aggregates information when the number of bidders becomes large. In their environment, the known number of bidders is exogenous and independent of the state of nature. They show the winning bid approaches the true value when the number of bidders becomes large if and only if there are unboundedly informative, favorable signals, $\frac{g_h(\overline{x})}{g_l(\overline{x})} = \infty$. If $\frac{g_h(\overline{x})}{g_l(\overline{x})} < \infty$ the limit equilibrium of the standard common value auction is partially revealing, but it becomes continuously more revealing as $\frac{g_h(\overline{x})}{g_l(\overline{x})}$ increases.[13]

In a related sequential search version of that model that differs mainly in that the seller searches sequentially for buyers, Lauermann and Wolinsky (2011) show that, when the search cost is negligible, nearly perfect information aggregation requires stronger conditions on the informativeness of the most favorable signals: Not only

[13] See our remarks on large ordinary common value auctions in Section 6.1. When $\bar{g} \to \infty$, inspection of the distribution of the winning bid, $\phi_w(\cdot|\rho, \bar{g}, r = 1)$, shows that distribution becomes degenerate with all its weight on v_h and v_l, respectively.

$\frac{g_h(\overline{x})}{g_l(\overline{x})} = \infty$, but also the likelihood ratio $\frac{g_h(x)}{g_l(x)}$ has to increase at a sufficiently fast rate when x approaches $\overline{x}$. If $\frac{g_h(\overline{x})}{g_l(\overline{x})} < \infty$, the equilibrium is complete pooling and both types trade at a price equal to the ex ante expected value.[14]

The present model combines elements from both of these environments. It is an auction in which the buyers compete directly in prices, but the endogenous state dependent solicitation of buyers is reminiscent of the search model. Indeed, in terms of information aggregation, the current model exhibits both patterns of information aggregation. The partially revealing equilibria resemble the equilibria of the standard auction. When $\frac{g_h(\overline{x})}{g_l(\overline{x})}$ is large, the aggregation of information is nearly perfect. To see this, recall that $r^*(\rho, \overline{g})$ is the solution to $J(r; \rho, \overline{g}) = 0$.

Lemma 20 *$\lim_{\overline{g}\to\infty} \overline{g}r^*(\rho, \overline{g}) = \infty$.*

Proof: Inspection of $J(r; \rho, \overline{g}) = \int_0^1 \left(x - \frac{1}{\overline{g}}\right) x^{\frac{1}{\overline{g}r-1}} \frac{\ln x}{(1+x\rho\overline{g}r)^2} dx$ reveals that if $\overline{g}r$ is bounded, then $J(r; \rho, \overline{g}) < 0$ for large $\overline{g}$. Therefore, it must be that $\overline{g} \to \infty$ implies $\overline{g}r^*(\rho, \overline{g}) \to \infty$ ∎

Now, it can be observed from Equation (11) that, when $\overline{g}r^*$ becomes large, the distribution of the winning bid, $\bar{F}_w(\cdot|\rho, \overline{g})$, puts almost all its weight on v_w. Thus, large $\overline{g}$ implies nearly perfect information aggregation in the partially separating equilibrium.

In contrast, the pooling equilibrium of Section 7.1 aggregates no information—the winning bid is at or below the ex ante expected value and such equilibria may exist independently of how large is $\frac{g_h(\overline{x})}{g_l(\overline{x})}$. The pooling equilibrium fails to aggregate information, similar to the equilibrium of the sequential search model. In fact, in the limit, the price of the auction may be strictly below the ex ante expected value, so that the high value seller receives even less revenue.

8.2 Signaling: Observable Number of Bidders

If the informed seller can disclose the number of solicited bidders verifiably, this number may signal the seller's information.[15] Specifically, consider a variation of our model in which the buyers observe the total number of solicited bidders before

[14] In Lauermann and Wolinsky (2011), the roles of buyers and sellers are reversed, so that the buyer is the informed and the sellers are the uninformed agents.

[15] Our interest is in analysing a specific trading environment in which the seller cannot verifiable communicate the number of solicited sellers. We discuss this variation as an exercise to provide further insight into the mechanism of the model. Moreover, our analysis illustrates the importance of the (un-)observability of the sampling strategy of the seller for the trading outcome. One may also be interested in the comparison with outcomes of the trading mechanism that is (ex-ante) optimal for the seller. This mechanism likely resembles a full-rent extraction mechanism as in Cremer and McLean (1988), because the seller may utilize the correlation of buyers' signals, with the added difficulty that the seller has private information.

submitting their bids, while everything else remains unchanged. This variation has two types of pure equilibria, a separating and a pooling equilibrium. In the pooling equilibrium, both types of the seller solicit the same number of bidders. Multiple pooling equilibria can be supported by specifying that buyers believe that a seller who solicits an out-of-equilibrium number of bidders must be of the low type, consequently bidding at most v_l. Bidding in the pooling equlibria is the same as in the standard common value auction, because the number of bidders is independent of the state. In the separating equilibrium, the low type solicits two bidders, and the high type solicits $n_h > 2$ bidders. Buyers bid v_l if two bidders are solicited and buyers bid v_h if n_h bidders are solicited. To ensure incentive compatibility, it must be that $v_h - n_h s = v_l - 2s$.[16] Thus, in the separating equilibrium, the payoff of each type of the seller is $v_l - 2s$. Depending on the equilibrium selection, the seller may be worse off if the number of solicited bidders can be disclosed verifiably than if it cannot, as in our original model.

A similar signal possibility arises in a variation where the seller can set an observable reservation price, so that the reservation price becomes a signal of the seller's information. Cai, Riley and Ye (2007) and Jullien and Mariotti (2006) study signaling through reservation prices in common value auctions. Jullien and Mariotti characterize both separating and pooling equilibria, while Cai, Riley and Ye appeal to the "intuitive criterion" to select the separating equilibrium.

8.3 Simultaneous Search

We discuss our model using the terminology of the auction literature. However, our model could equivalently be thought of as a simultaneous search model along the lines of Burdett and Judd (1983). It adds adverse selection to their model. The seller in our model is the counterpart of the buyer in their model. Their buyer collects a sample of prices and then proceeds to purchase. So does the auctioneer in our model. The roles of the seller and the buyers in our model can, of course, be reversed to make the models exactly parallel. The important difference is in the private information that the sampling agent has in our model. The private information implies both additional substantive insights and some additional analytical challenges. Together, the current paper and Lauermann and Wolinsky (2011) span the two common modes of search, sequential and simultaneous.

[16] There are additional partially separating equilibria in mixed strategies. For example, the high type may randomly choose either 2 or n_h bidders. If 2 bidders are chosen, bidders bid as in the corresponding common value auction in which the priors are adjusted appropriately for the seller's strategy.

8.4 About the Assumptions

The assumption that the state is binary was used in the proof of the monotonicity of the bidders' best response. In this sense, the assumption plays an important role. Nevertheless, the assumption buys us more than we need—the best response to any strategy is monotone. Therefore, it is possible that we are still able to obtain similar results for monotone equilibria with more than two states. This seems to be an interesting extension that continued work on this subject may address.

The boundedness of the signal likelihood ratio, $\frac{g_h(\bar{x})}{g_l(\bar{x})} < \infty$, is important for our characterization argument. It is not clear that the same analysis can be carried out when signals are unboundedly informative.

We have assumed that the seller is fully informed about the state. It is hard to see that anything substantial would change if the seller observed a noisy signal of the state instead. Of course, if the signal were not binary, then the model would be like a multi-state world.

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