# Week 3 Quiz: Differential Calculus: The Derivative and Rules of Differentiation

SGPE Summer School 2016

## Limits

**Question 1:** Find $\lim_{x \to 3} f(x)$:

$$f(x) = \frac{x^2 - 9}{x - 3}$$

(A) $+\infty$

(B) -6

(C) 6

(D) Does not exist!

(E) None of the above

**Answer:** (C) Note the the function $f(x) = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x + 3$ is actually a line. However it is important to note the this function is *undefined* at $x = 3$. Why? $x = 3$ requires dividing by zero (which is inadmissible). As $x$ approaches 3 from below and from above, the value of the function $f(x)$ approaches $f(3) = 6$. Thus the limit $\lim_{x \to 3} f(x) = 6$.

**Question 2:** Find $\lim_{x \to 2} f(x)$:

$$f(x) = 1776$$

(A) $+\infty$

(B) 1770

(C) $-\infty$

(D) Does not exist!

(E) None of the above

**Answer:** (E) The limit of any constant function at any point, say $f(x) = C$, where $C$ is an arbitrary constant, is simply $C$. Thus the correct answer is $\lim_{x \to 2} f(x) = 1776$.

**Question 3:** Find $\lim_{x \to 4} f(x)$:

$$f(x) = ax^2 + bx + c$$

(A) $+\infty$

(B) 16a + 4b + c

(C) $-\infty$

(D) Does not exist!

(E) None of the above

**Answer:** (B) Applying the rules of limits:

$$\begin{aligned}\lim_{x\to 4} ax^2 + bx + c &= \lim_{x\to 4} ax^2 + \lim_{x\to 4} bx + \lim_{x\to 4} c \\ &= a\left[\lim_{x\to 4} x\right]^2 + b\lim_{x\to 4} x + c \\ &= 16a + 4b + c\end{aligned}$$

**Question 4:** Find the limits in each case:

(i) $\lim_{x\to 0} \frac{x^2}{|x|}$

(ii) $\lim_{x\to 3} \frac{2x+3}{4x-9}$

(iii) $\lim_{x\to 6} \frac{x^2-3x}{x+3}$

**Answer:** $(i)$ $\lim_{x\to 0} \frac{x^2}{|x|} = \lim_{x\to 0} \frac{(|x|)^2}{|x|} = \lim_{x\to 0} \mid x \mid = 0$

$(ii)$ $\lim_{x\to 3} \frac{2x+3}{4x-9} = \frac{2\cdot 3+3}{4\cdot 3-9} = 3$

$(iii)$ $\lim_{x\to 6} \frac{x^2-3x}{x+3} = \frac{6^2-3\cdot 6}{6+3} = 2$

**Question 5:** Show that $\lim_{x\to 0} \sin x = 0$ (Hint: $-x \leq \sin x \leq x$ for all $x \geq 0$.)

**Answer:** Given hint and squeeze theorem we have $\lim_{x\to 0} -x = 0 \leq \lim_{x\to 0} \sin x \leq 0 = \lim_{x\to 0} x$ hence, $\lim_{x\ to 0} \sin x = 0$

**Question 6:** Show that $\lim_{x\to 0} x \sin(\frac{1}{x}) = 0$

**Answer:** Note first that for any real number $t$ we have $-1 \leq \sin t \leq 1$ so $-1 \leq \sin(\frac{1}{x}) \leq 1$. Therefore, $-x \leq x\sin(\frac{1}{x}) \leq x$ and by squeeze theorem $\lim_{x\to 0} x \sin \frac{1}{x} = 0$.

# Continuity and Differentiability

**Question 7:** Which of the following functions are *NOT* everywhere continuous:

(A) $f(x) = \frac{x^2-4}{x+2}$

(B) $f(x) = (x+3)^4$

(C) $f(x) = 1066$

(D) $f(x) = mx + b$

(E) None of the above

**Answer:** (A) Remember that, informally at least, a *continuous* function is one in which there are no breaks its curve. A continuous function can be drawn without lifting your pencil from the paper. More formally, a function $f(x)$ is *continuous* at the point $x = a$ if and only if:

1. $f(x)$ is defined at the point $x = a$,
2. the limit $\lim_{x\to a} f(x)$ exists,
3. $\lim_{x\to a} f(x) = f(a)$

The function $f(x) = \frac{x^2-4}{x+2}$ is not everywhere continuous because the function is not defined at the point $x = -2$. It is worth noting that $\lim_{x\to -2} f(x)$ does in fact exist! **The existence of a limit at a point does not guarantee that the function is continuous at that point!**

**Question 8:** Which of the following functions are continuous:

(A) $f(x) = |x|$

(B) $f(x) = \begin{cases} 3 & x < 4 \\ \frac{1}{2}x + 3 & x \geq 4 \end{cases}$

(C) $f(x) = \frac{1}{x}$

(D) $f(x) = \begin{cases} \ln x & x < 0 \\ 0 & x = 0 \end{cases}$

(E) None of the above

**Answer:** (A) The absolute value function $f(x) = |x|$ is defined as:

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Does this function satisfy the requirements for continuity? Yes! The critical point to check is $x = 0$. Note that the function is defined at $x = 0$; the $\lim_{x\to 0} f(x)$ exists; and that $\lim_{x\to 0} f(x) = 0 = f(0)$.

**Question 9:** Which of the following functions are *NOT* differentiable:

(A) $f(x) = |x|$

(B) $f(x) = (x + 3)^4$

(C) $f(x) = 1066$

(D) $f(x) = mx + b$

(E) None of the above

**Answer:** (A) Remember that continuity is a *necessary* condition for differentiability (i.e., every differentiable function is continuous), but continuity is not a *sufficient* condition to ensure differentiability (i.e., not every continuous function is differentiable). Case in point is $f(x) = |x|$. This function is in fact continuous (see previous question). It is not however differentiable at the point $x = 0$. Why? The point $x = 0$ is a cusp (or kink). There are an infinite number of lines that could be tangent to the function $f(x) = |x|$ at the point $x = 0$, and thus the derivative of $f(x)$ would have an infinite number of possible values.

**Question 10:** Is function

$$f(x) = \begin{cases} 0 & : x = 0 \\ x \sin(1/x) & : x \neq 0 \end{cases}$$

continuous at point 0?

**Answer:** Note that $f$ is continuous at a point $a$ if

$$\lim_{x\to a} f(x) = f(\lim_{x\to a} x).$$

In this case, we take $a = 0$ and

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin(1/x) = 0$$

by question 6. Moreover,

$$f(\lim_{x\to 0} x) = f(0) = 0$$

thus, $f$ is continuous at 0.

# Derivatives

**Question 11:** Find the derivative of the following function:

$$f(x) = 1963$$

(A) $+\infty$

(B) 1963

(C) $-\infty$

(D) 0

(E) None of the above

**Answer:** (D) The derivative of a constant function is always zero.

**Question 12:** Find the derivative of the following function:

$$f(x) = x^2 + 6x + 9$$

(A) $f'(x) = 2x + 6 + 9$

(B) $f'(x) = x^2 + 6$

(C) $f'(x) = 2x + 6$

(D) $f'(x) = 2x$

(E) None of the above

**Answer:** (C) Remember that 1) the derivative of a sum of functions is simply the sum of the derivatives of each of the functions, and 2) the power rule for derivatives says that if $f(x) = kx^n$, then $f'(x) = nkx^{n-1}$. Thus $f'(x) = 2x^{2-1} + 6x^{1-1} + 0 = 2x + 6$.

**Question 13:** Find the derivative of the following function:

$$f(x) = x^{\frac{1}{2}}$$

(A) $f'(x) = -\frac{1}{2\sqrt{x}}$

(B) $f'(x) = \frac{1}{\sqrt{x}}$

(C) $f'(x) = \frac{1}{2\sqrt{x}}$

(D) $f'(x) = \sqrt{x}$

(E) None of the above

**Answer:** (C) Remember that the power rule for derivatives works with fractional exponents as well! Thus $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

**Question 14:** Find the derivative of the following function:

$$f(x) = 5x^2(x + 47)$$

(A) $f'(x) = 15x^2 + 470x$

(B) $f'(x) = 5x^2 + 470x$

(C) $f'(x) = 10x$

(D) $f'(x) = 15x^2 - 470x$

(E) None of the above

**Answer:** (A) Ideally, you would solve this problem by applying the product rule. Set $g(x) = 5x^2$ and $h(x) = (x + 47)$, then $f(x) = g(x)h(x)$. Apply the product rule:

$$\begin{aligned} f'(x) &= g'(x)h(x) + g(x)h'(x) \\ &= 10x(x + 47) + 5x^2(1) \\ &= 10x^2 + 470x + 5x^2 \\ &= 15x^2 + 470x \end{aligned}$$

**Question 15:** Find the derivative of the following function:

$$f(x) = \frac{5x^2}{x + 47}$$

(A) $f'(x) = \frac{5x^2-470x}{(x+47)^2}$

(B) $f'(x) = \frac{10x^2+470x}{(x+47)}$

(C) $f'(x) = 10x$

(D) $f'(x) = \frac{5x^2+470}{(x+47)^2}$

(E) None of the above

**Answer:** (A) Ideally, you would solve this problem by applying the quotient rule. Set $g(x) = 5x^2$ and $h(x) = (x + 47)$, then $f(x) = \frac{g(x)}{h(x)}$. Apply the quotient rule:

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \\ &= \frac{10x(x + 47) - 5x^2(1)}{(x + 47)^2} \\ &= \frac{10x^2 + 470x - 5x^2}{(x + 47)^2} \\ &= \frac{5x^2 + 470x}{(x + 47)^2} \end{aligned}$$

**Question 16:** Find the derivative of the following function:

$$f(x) = 5(x + 47)^2$$

(A) $f'(x) = 15x^2 + 470x$

(B) $f'(x) = 10x - 470$

(C) $f'(x) = 10x + 470$

(D) $f'(x) = 15x^2 - 470x$

(E) None of the above

**Answer:** (C) Ideally, you would solve this problem by applying the chain rule. Set $g(h) = 5h^2$ and $h(x) = (x + 47)$, then $f(x) = g(h(x))$. Apply the chain rule:

$$\begin{aligned} f'(x) &= g'(h)h'(x) \\ &= 10h \\ &= 10(x + 47) \\ &= 10x + 470 \end{aligned}$$

# Higher Order Derivatives

**Question 17:** Find the second derivative of the following function:

$$f(x) = 5x^2(x+47)$$

(A) $f''(x) = 30x - 470$

(B) $f''(x) = 30x + 470$

(C) $f''(x) = 15x^2 + 235$

(D) $f''(x) = 15x^2 + 470x$

(E) None of the above

**Answer:** (B) The second derivative is just the derivative of the first derivative. Simplest solution would be to multiply to re-write the function as $f(x) = 5x^2(x+47) = 5x^3 + 235x^2$. Now take the derivative: $f'(x) = 15x^2 + 470x$. Taking the derivative again yields the second derivative: $f''(x) = 30x + 470$.

**Question 18:** Find the third derivative of the following function:

$$f(x) = 5x^2(x+47)$$

(A) 15

(B) $15 + x$

(C) $30x$

(D) $30x + 470$

(E) None of the above

**Answer:** (E) Just take the derivative of your answer to Question 12 to get the third derivative of $f(x) = 5x^2(x+47)$. Answer: $f'''(x) = 30$.

**Question 19:** Suppose that you have the following utility function:

$$u(x) = \sqrt{x}$$

Find $-\frac{u''(x)}{u'(x)}$.

(A) $\frac{1}{2x}$

(B) $-\frac{1}{2x}$

(C) $2x$

(D) $-2x$

(E) None of the above

**Answer:** (A) The ratio $-\frac{u''(x)}{u'(x)}$ is called the Arrow-Pratt measure of relative risk aversion and you will encounter it in core microeconomics. The first derivative of the utility function (otherwise known as marginal utility) is $u'(x) = \frac{1}{2\sqrt{x}}$ (see Question 9 above). The second derivative is $u''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4\sqrt{x^3}}$. Thus the Arrow-Pratt measure of relative risk aversion is:

$$-\frac{u''(x)}{u'(x)} = -\frac{-\frac{1}{4\sqrt{x^3}}}{\frac{1}{2\sqrt{x}}} = \frac{2\sqrt{x}}{4\sqrt{x^3}} = \frac{1}{2x}$$